

Floer Homology for Symplectomorphism

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Abstract. Let (M, ω) be a compact symplectic manifold, and ϕ be a symplectic diffeomorphism on M , we define a Floer-type homology $FH_*(\phi)$ which is a generalization of Floer homology for symplectic fixed points defined by Dostoglou and Salamon for monotone symplectic manifolds. These homology groups are modules over a suitable Novikov ring and depend only on ϕ up to a Hamiltonian isotopy.

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1 Introduction.

Floer homology for symplectic manifolds is a great by-product for proving the famous Arnold conjectures[A] concerning about the number of non-degenerate fixed points of Hamiltonian diffeomorphisms of any compact symplectic manifold and the number of transversal intersection points of any Lagrangian submanifold with its Hamiltonian deformations. For the first case, since the fixed points of a Hamiltonian diffeomorphism ψ_H correspond to the time-1 periodic solutions of the Hamiltonian vector field X_t generated by a time-dependent Hamiltonian function H_t , the problem is equivalent to estimating the number of non-degenerate periodic solutions. To do this, A. Floer[F1][F2][F3] initialed the method of using Gromov's pseudoholomorphic curves as connecting orbits between the periodic solutions and, by counting connecting orbits, establishing infinite dimensional Morse-Witten complex and homology. For monotone (by Floer[F3]) and semi-positive (by Hofer-Salamon[HS], Ono[On1]) symplectic manifolds, defining Floer homology only involves the moduli space of connecting orbits, and the Gromov's weak compactness for J -holomorphic curves can be proved (with some kind of dimension counting argument). As for general symplectic manifolds, however, the phenomenon of bubbling-off spheres can not be avoided by transversality arguments, and for this reason, the Gromov's compactness for the moduli space of J -holomorphic curves connecting periodic solutions can not always hold. By considering the enlarged moduli space of stable maps and using the elaborate virtual techniques (or establishing the so-called Kuranishi structure), Fukaya-Ono[FO], Liu-Tian[LT1] independently gave the definition of general Floer homology. As a result, the general Arnold conjecture for nondegenerate fixed points of Hamiltonian diffeomorphism was proved.

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By now, there exist some variants of Floer-type homology. For instance, for (simply connected) monotone symplectic manifold (M, ω) , using the same method by Floer[F3] with some modified details, Dostoglou-Salamon[DS2] defined a new (or generalized) version of Floer homology for symplectic fixed points of some special symplectomorphism as $\phi_H = \psi_1^{-1} \circ \phi$, where ψ_1 is the time-1 map of symplectomorphisms $\psi_t : M \rightarrow M$ generated by a Hamiltonian H_t satisfying $H_t = H_{t+1} \circ \phi$, *i.e.*

$$\frac{d}{dt}\psi_t = X_t, \quad \psi_0 = id, \quad \iota(X_t)\omega = dH_t$$

with $\psi_{t+1} \circ \phi_H = \phi \circ \psi_t$, and ϕ is a given symplectomorphism on M . It is easy to see that when $\phi = id$, ϕ_H is a Hamiltonian diffeomorphism, and the generalized Floer homology is just the ordinary one. For a symplectomorphism $\phi \in \text{Symp}_0(M, \omega)$, *i.e.* in the identity component of symplectic diffeomorphism group $\text{Symp}(M, \omega)$, Ono[On2] also defined another version of so-called Floer-Novikov homology for general symplectic manifold.

In the present paper, we generalize the Dostoglou-Salamon's construction to general symplectic manifold and define such Floer homology. The fundamental idea is to study the moduli space of stable connecting orbits from open surface from which we construct virtual cycle, and to use it to define a complex which is a module over a Novikov ring generated by the solutions of the Hamiltonian equation above, and a suitable boundary operator on it whose homology is the favorite one.

Firstly, we review the basic idea of dealing with problems of moduli space.

Definition 1.1 *We say a triple $(\mathcal{E}, \mathcal{B}, s)$ is a Fredholm system if the following hold*

- 1) \mathcal{E}, \mathcal{B} are two smooth Banach orbifolds, $\pi : \mathcal{E} \rightarrow \mathcal{B}$ is a Banach orbibundle, whose each fibre \mathcal{E}_x , $x \in \mathcal{B}$ is a Banach space.
- 2) $s : \mathcal{B} \rightarrow \mathcal{E}$ is a smooth section, and for $\forall x \in \mathcal{B}$ the linearization of s at x

$$D_x s : T_x \mathcal{B} \rightarrow \mathcal{E}_x$$

is a Fredholm operator and $s^{-1}(0)$ is compact. s is called a Fredholm section.

And the zero set of section $\mathcal{M} = s^{-1}(0)$ is called the moduli space of the system.

In particular, if the linearization map $D_x s$ is surjective for all $x \in \mathcal{M}$, then \mathcal{M} is a smooth orbifold and the dimension of \mathcal{M} is equal to the Fredholm index of $D_x s$. If $\text{ind}(D_x s) = d$, then \mathcal{M} can be considered as a cycle in $H_d(\mathcal{B})$ representing the Euler class of the bundle $\mathcal{E} \rightarrow \mathcal{B}$. Thus some invariants (for example G-W invariants), can be defined for any cohomology form $\alpha \in H^d(\mathcal{B}, \mathbb{R})$ as $\Phi(\alpha) = \int_{\mathcal{M}} \alpha$.

Actually, defining Floer homology for fixed point of Hamiltonian diffeomorphism Φ_H is also based on the study of thus moduli space. For instance, let $\mathcal{B} = \mathcal{W}^{1,p}(u^*TM)$ be some suitable completion of the space of smooth maps $u : \Sigma \rightarrow (M, \omega)$ with some suitable boundary condition, where Σ is a Riemann surface (for example $\Sigma = \mathbb{R} \times S^1$ and $\lim_{s \rightarrow \pm\infty} u(s, t) = x_{\pm}(t)$ are periodic solutions of the Hamiltonian equation), and let \mathcal{E} be the Banach bundle over \mathcal{B} , whose fiber at $u \in \mathcal{B}$ is the Banach space $L^p(\Lambda^{0,1}(u^*TM))$, then the $\bar{\partial}_J$ map or its some perturbation $\bar{\partial}_{J,H}$ can be thought as a section of bundle $\mathcal{E} \rightarrow \mathcal{B}$. In some ideal case, for instance M is a monotone symplectic manifold, Floer proved that $(\mathcal{E}, \mathcal{B}, \bar{\partial}_{J,H})$ is a Fredholm system,

and for generic chosen pair (J, H) , the linearization operator D_u of $\bar{\partial}_{J,H}$ is surjective at any $u \in \mathcal{M} = \bar{\partial}_{J,H}^{-1}(0)$. Using this moduli space \mathcal{M} , especially, by studying its 1 and 2 dimensional components, Floer defined his well known homology.

However, in general, the moduli space \mathcal{M} is not compact. Floer's method for defining homology for general symplectic manifold is invalid, since we can not naturally get a Fredholm system $(\mathcal{E}, \mathcal{B}, \bar{\partial}_{J,H})$ as above. In fact, one can overcome this difficult by considering a larger space of maps, *i.e.* the stable compactification $\overline{\mathcal{M}}$, say moduli space of stable maps, which is the zero set of an enlarged bundle, introduced by Kontsevich. At the same time, although the new enlarged moduli space is compact, the appearing of multiple covered J -spheres with negative Chern number would make the linearization map $D_x s$ be not always surjective for any $x \in \overline{\mathcal{M}}$. This also makes the dimension of the "boundary" of $\overline{\mathcal{M}}$ too larger than estimated by index theorem. The idea of virtual techniques to deal with this problem is to construct generic perturbative section $\bar{\partial}_{J,H} + \nu$ of some orbibundle (or, say multi-bundle) over the enlarged space of stable $\mathcal{W}^{1,p}$ -maps, with the virtual moduli space \mathcal{M}^ν as the zero set of this perturbed section, and to construct the so-called virtual moduli cycle $C(\mathcal{M}^\nu)$, from which one can derive the well-defined Floer-type homology.

Now in our setting, for any two \tilde{x}_-, \tilde{x}_+ the stable moduli space is $PM(\tilde{x}_-, \tilde{x}_+)$, where we denote $\tilde{x}_-(\tilde{x}_+)$ for the lift of fixed point $x_-(x_+)$ of ϕ_H in some universal covering space. Roughly speaking, $PM(\tilde{x}_-, \tilde{x}_+)$ consists of all stable (J, H) -orbits connecting \tilde{x}_- and \tilde{x}_+ , such a stable connecting orbit $V : \Sigma \rightarrow M$ contains some main components v_m , $m = 1, \dots, K$, which each is a $\bar{\partial}_{J,H}$ -orbit, and some bubble components f_b , $b = 1, \dots, L$, which each is a J -holomorphic sphere. And $PM(\tilde{x}_-, \tilde{x}_+)$ is the natural compactification of the ordinary moduli space $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ which consists of only $\bar{\partial}_{J,H}$ -orbits with open domain \mathbb{R}^2 . We refer the reader to section 2 and 3 for related definitions. For monotone symplectic manifold M , Dostoglou-Salamon used only the moduli space $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ to define the homology for symplectomorphism ϕ , while for general symplectic manifold we must consider its stable compactification.

Then the ambient space is denoted by $\mathcal{B}(\tilde{x}_-, \tilde{x}_+)$, and in the partially smooth category (c.f. [M2] or section 4) we show that there exists a neighborhood \mathcal{W} of $PM(\tilde{x}_-, \tilde{x}_+)$ in $\mathcal{B}(\tilde{x}_-, \tilde{x}_+)$ with the so-called multi-fold structure or atlas $\tilde{\mathcal{V}}$, and we define a multi-bundle $\tilde{\mathcal{E}}$ over it, then we can show that in a small covering neighborhood \mathcal{W}_ϵ , locally the generic perturbed map $\bar{\partial}_{J,H} + \nu$ gives the "fine" section, and all of them fit together to give a multi-section \tilde{s} of this bundle, In other words, for generic pair (J, H) we can obtain a transverse Fredholm system $(\tilde{\mathcal{E}}, \tilde{\mathcal{V}}, \mathcal{W}, \tilde{s})$ with index d , and the zero set of this multi-section is the virtual moduli space, denoted by $PM^\nu(\tilde{x}_-, \tilde{x}_+)$ which is compact and can be considered as a relative cycle with correct dimension d estimated by index theorem.

We then define a graded \mathbb{Q} -space $C_* = C_*(J, H, \phi) = \oplus_n C_n(J, H, \phi)$ as usual. Simply speaking, we can define a functional F on some covering of a path space, and $C_n(J, H, \phi)$ is generated by the critical points of F with Conley-Zehnder index n . And C_* is in general an infinite dimensional space over \mathbb{Q} but a finite dimensional space over a Novikov ring $\Lambda_{\omega, \phi}$ (c.f. section 6.2 for details). Then we can just define

the boundary operator by $\delta_{J,H,\nu} : C_n \rightarrow C_{n-1}$, such that for any $\tilde{x} \in \text{Crit}_n(F)$,

$$\delta_{J,H,\nu}(\tilde{x}) = \sum_{\tilde{y} \in \text{Crit}_{n-1}(F)} \#(PM^\nu(\tilde{y}, \tilde{x}))\tilde{y},$$

where $\text{Crit}_n(F)$ denotes the set of critical points of F with Conley-Zehnder index n . Now we can state our main result

Theorem 1 *Let (M, ω) be a compact symplectic manifold with compatible almost complex structure J , and ϕ be a given symplectomorphism on M . Then for a generic pair of J and time dependent Hamiltonian function $H : \mathbb{R} \times M \rightarrow \mathbb{R}$ satisfying $H_t = H_{t+1} \circ \phi$, we can construct a compact relative rational cycle PM^ν and a boundary operator $\delta_\nu = \delta_{J,H,\nu} : C_* \rightarrow C_*$, such that $\delta_\nu^2 = 0$. So the Floer homology $FH_*(J, H, \phi, \nu)$ is the homology of this chain complex (C_*, δ_ν) . Moreover, Floer homology $FH_*(J, H, \phi, \nu)$ is independent of the choice of the generic pair (J, H) , and it depends on the symplectomorphism ϕ only up to Hamiltonian isotopy, i.e. there exists a natural isomorphism*

$$FH_*(J_0, H_0, \phi_0, \nu_0) \rightarrow FH_*(J_1, H_1, \phi_1, \nu_1),$$

provided ϕ_0 and ϕ_1 are related by a Hamiltonian isotopy.

Remark. With the Floer homology $FH_*(\phi)$ defined above, we can consider the pair-of-pants construction suggested by Donaldson, i.e. we can consider a homomorphism

$$FH_*(\psi) \otimes FH_*(\phi) \rightarrow FH_*(\psi \circ \phi).$$

If $\psi = id$, this induces the quantum cap product. We will not study this topic in the present article, the author plans to treat the quantum cap product and its related applications in another paper.

In the present paper, we also use the similar virtual techniques with some modifications based on Liu-Tian's work [LT1] to construct Floer homology of symplectomorphism for general symplectic manifold. In fact, the methods by Fukaya-Ono[FO] and Li-Tian [LiT] can be also used for this purpose, however, we want to use as less analytic tools as possible at the cost of more topological arguments. Also the technique used by Ruan [R] may be applied for this by generalizing arguments in [Si] to construct virtual neighborhoods.

The contents of the paper are as follows. In section 2 we define the ordinary moduli space $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ of $\bar{\partial}_{J,H}$ -connecting orbits with the variant in that the solutions of the Hamiltonian equation given above, whose lifts are considered as the critical points of action functional, are unnecessary periodic ones. In section 3 we define the stable connecting orbits and the enlarged stable moduli space $PM(\tilde{x}_-, \tilde{x}_+)$. Then in section 4 we show that this enlarged space is Hausdorff and compact, and we study its small neighborhood in its ambient space. In section 5, in a more abstract setting and in the partially smooth category, we show the concepts of multi-fold, multi-bundle and multi-section which can be found (maybe with slight difference in different settings) in many literatures (cf. [FO][LT1][M2][S][Si]), and show the general method of constructing virtual cycle. In section 6 we prove that locally we can

get a transverse Fredholm system by generically perturbation. As an application of the results in preceding sections, in section 7 we construct the virtual moduli space and the relative virtual moduli cycle $P\mathcal{M}'$ and define the Floer-type homology.

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2 Moduli space of connecting orbits.

We take the twisted free loop space as

$$\Omega_\phi = \{\gamma : \mathbb{R} \rightarrow M : \gamma(t+1) = \phi(\gamma(t))\}.$$

And consider the closed 1-form on Ω_ϕ

$$\langle \alpha(\gamma), \xi \rangle = \int_0^1 \omega(\dot{\gamma} - X_t(\gamma), \xi) dt.$$

We say that an almost complex structure $J : TM \rightarrow TM$ is compatible with symplectic form ω if they induce a Riemannian metric $g(\xi, \eta) = \omega(\xi, J\eta)$. And we denote the set of all compatible almost complex structure by $\mathcal{J}(M, \omega)$. Choose a smooth map $\mathbb{R} \rightarrow \mathcal{J}(M, \omega) : t \mapsto J_t$ such that $J_t = \phi^* J_{t+1}$. Such a structure determines a metric on Ω_ϕ .

Now, we consider the minimal covering $\pi : \tilde{\Omega}_\phi \rightarrow \Omega_\phi$ such that the form $\pi^*\alpha$ is exact, *i.e.* there is a functional F on $\tilde{\Omega}_\phi$, such that $\pi^*\alpha = dF$, and its structure group Γ is free abelian. In general, we may additionally assume that there is an injective homomorphism $i : \Gamma \rightarrow \pi_2(M)$. For instance, if the symplectic manifold M is simply connected, then $\Gamma = \pi_1(\Omega_\phi) \cong \pi_2(M)$.

In this article, we will at first consider a connected component of Ω_ϕ , and describe a certain covering space of it. Firstly, we choose and fix a path $\gamma_0 \in \Omega_\phi$, and consider the component of Ω_ϕ containing γ_0 , denote it by $\Omega_\phi^{\gamma_0}$ or Ω_ϕ^0 . We denote by $C([0, 1] \times \mathbb{R}, M)$ for all the continuous maps $w : [0, 1] \times \mathbb{R} \rightarrow M$, we consider the set of pairs as

$$\{(\gamma, w) \mid w(0, \cdot) = \gamma_0, w(1, \cdot) = \gamma, \gamma \in \Omega_\phi^0, w \in C([0, 1] \times \mathbb{R}, M)\}.$$

We define a homomorphism $\mathcal{I}_\omega : \pi_2(M) \rightarrow \mathbb{R}$, $\mathcal{I}_\omega(A) = \int_A \omega$, $\forall A \in \pi_2(M)$. And we define a weaker equivalence relation by

$$(\gamma, w) \sim (\gamma', w') \iff \gamma = \gamma' \text{ and } \int_{[0,1] \times [0,1]} w^* \omega = \int_{[0,1] \times [0,1]} (w')^* \omega.$$

The universal covering space of Ω_ϕ^0 can be defined as the set of equivalence classes of the pairs defined above, denote it by $\tilde{\Omega}_\phi^0 = \{[\gamma, w]\}$. Then the Γ is the group of deck transformation of the covering

$$\tilde{\Omega}_\phi^0 \rightarrow \Omega_\phi^0.$$

One can check that this is a regular covering, say, $\tilde{\Omega}_\phi^0/\Omega_\phi^0 \simeq \Gamma$ for any fixed $\gamma_0 \in \Omega_\phi^0$. Furthermore, we see that $\pi_2(M)$ acts on $\tilde{\Omega}_\phi$ via $\tilde{\gamma} = [\gamma, w] \rightarrow [\gamma, A\#w] = \tilde{\gamma}\#A$, where $A\#w$ denotes the equivalence class of the connected sum $a\#w$ for any representative a of A . The covering map is $\pi : \tilde{\Omega}_\phi^0 \rightarrow \Omega_\phi^0$, $\pi(\tilde{\gamma}) = \gamma$. Now, we can write the functional $F_{\gamma_0} : \tilde{\Omega}_\phi^0 \rightarrow \mathbb{R}$ by

$$F_{\gamma_0}([\gamma, w]) = \int_{[0,1] \times [0,1]} w^* \omega - \int_0^1 H(\gamma(t), t) dt.$$

Then the universal cover $\tilde{\Omega}_\phi$ of Ω_ϕ , and the functional F on $\tilde{\Omega}_\phi$ can be defined componentwise. The critical points of F are such $[\gamma, w]$ with γ being the solution of Hamiltonian equation listed in the introduction.¹

In the sequel, for simplicity of notations, we will just write Ω_ϕ and $\tilde{\Omega}_\phi$ for Ω_ϕ^0 and $\tilde{\Omega}_\phi^0$, respectively, if without the danger of confusion. We also use \tilde{x} to denote a critical point of the functional F , which can be considered as a lift of the symplectic path $\psi_t(x)$ or a lift of a fixed point x of the symplectomorphism ϕ_H .

The gradient ∇F of the functional F , with respect to the lift of the metric on Ω_ϕ , is a Γ -invariant vector field on $\tilde{\Omega}_\phi$, and $\pi_* \nabla F = \text{grad}_\alpha$. Then we consider the moduli space of thus gradient flows connecting a pair of critical points $(\tilde{x}_-, \tilde{x}_+)$ of F

$$\widehat{\mathcal{M}}(\tilde{x}_-, \tilde{x}_+) = \left\{ \tilde{u} : \mathbb{R} \rightarrow \tilde{\Omega} \mid \begin{array}{l} \frac{d\tilde{u}(s)}{ds} = -\nabla F(\tilde{u}(s)), \tilde{u} \text{ is not constant,} \\ \lim_{s \rightarrow \pm\infty} \tilde{u}(s) = \tilde{x}_\pm \end{array} \right\}.$$

It is also called the moduli space of connecting orbits. Denote the collection by $\widehat{\mathcal{M}} = \bigcup_{\tilde{x}_\pm} \widehat{\mathcal{M}}(\tilde{x}_-, \tilde{x}_+)$, the non-parameterized space by

$$\mathcal{M}(\tilde{x}_-, \tilde{x}_+) = \widehat{\mathcal{M}}(\tilde{x}_-, \tilde{x}_+)/\mathbb{R},$$

and the natural quotient map by $q : \widehat{\mathcal{M}} \rightarrow \mathcal{M}$. For generic pair (H, J_t) , all $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ are finite dimensional smooth manifolds.

The map $u : \mathbb{R}^2 \rightarrow M$, defined by $u(s, t) = \pi(\tilde{u}(s))(t)$, satisfies the following perturbed Cauchy-Riemann equation

$$\bar{\partial}_{J,H}(u) = \partial_s u + J_t(u)(\partial_t u - X_t(u)) = 0 \quad (1)$$

with boundary condition

$$u(s, t+1) = \phi(u(s, t)), \quad (2)$$

and has limits

$$\lim_{s \rightarrow \pm\infty} u(s, t) = \psi_t(x_\pm), \quad x_\pm = \phi_H(x_\pm), \quad (3)$$

where x_\pm are nondegenerate fixed points of ϕ_H .

¹For a special case $\phi = id$, our covering space and functional F are different from the usual covering space \tilde{L} and action functional a_H (e.g. [HS][LT1]), however, in the path space Ω_ϕ they have the same paths as the projection of the set of critical points. The author thinks it is interesting to consider the relation between the two kinds of Floer homology constructed under the two settings.

The energy of u is

$$E(u) = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 (|\partial_s u|^2 + |\partial_t u - X_t(u)|^2) dt ds < \infty$$

It is easy to see that $\tilde{x}_- \# u = [x_+, w \# u] = [x_+, w'] = \tilde{x}_+$, where $w' = w \# u$ is the obvious concatenation of w and u along the path x_- . And we have the equality

$$E(u) = F(\tilde{x}_+) - F(\tilde{x}_-). \quad (4)$$

For any smooth function $u : \mathbb{R}^2 \rightarrow M$ satisfying $u(s, t+1) = \phi(u(s, t))$, we denote $W_\phi^{k,p}(u^*TM)$ for the Sobolev completion of the space of smooth vector fields $\xi(s, t) \in T_{u(s,t)}M$ along u , which satisfy $\xi(s, t+1) = d\phi(u(s, t))\xi(s, t)$ and have compact support on $\mathbb{R} \times [0, 1]$, with respect to the $W^{k,p}$ -norm over $\mathbb{R} \times [0, 1]$. And denote $L_\phi^p(u^*TM) = W_\phi^{0,p}(u^*TM)$. Then for solution u of (1) and (2), we get the following linear operator by linearizing (1)

$$D_u : W_\phi^{1,p}(u^*TM) \rightarrow L_\phi^p(u^*TM)$$

defined by

$$D_u \xi = \nabla_s \xi + J_t(u)(\nabla_t \xi - \nabla_\xi X_t(u)) + \nabla_\xi J_t(u)(\partial_t u - X_t(u)),$$

where ∇ denotes the Levi-Civita connection associated to the metric $g_t(\cdot, \cdot) = \omega(\cdot, J_t \cdot)$. If the fixed points $x_\pm = \phi_H(x_\pm)$ are nondegenerate then D_u is a Fredholm operator and its index is given by the Maslov index of u .

More precisely, we follow [DS1] to show the definition of the Maslov index. Let $Sp(2n)$ be the group of symplectic matrices. And denote the singular subset by $Sing(2n) = \{A \in Sp(2n) \mid \det(Id - A) = 0\}$, which is called *Maslov cycle*, its complement is an open and dense subset of $Sp(2n)$, denote it by

$$Sp^*(2n) = Sp(2n) - Sing(2n).$$

For any path $\Psi : [0, 1] \rightarrow Sp(2n)$, with $\Psi(0) = Id$ and $\Psi(1) = Sp^*(2n)$, there exists the so-called Conley-Zehnder index $\mu_{CZ}(\Psi)$ (c.f. [RS][SZ]). Given two nondegenerate fixed points x_\pm of ϕ_H , denote by $\mathcal{P}(x_-, x_+)$ the space of all smooth functions $u : \mathbb{R}^2 \rightarrow M$ satisfying (2) and (3). For any $u \in \mathcal{P}(x_-, x_+)$, we choose a trivialization $\Phi(s, t) : \mathbb{R}^{2n} \rightarrow T_{u(s,t)}M$ such that

$$\Phi(s, t)^* \omega = \omega_0, \quad \Phi(s, t+1) = d\phi(u(s, t))\Phi(s, t)$$

and $\lim_{s \rightarrow \pm\infty} \Phi(s, t) = \Phi^\pm(t) : \mathbb{R}^{2n} \rightarrow T_{\psi_t(x_\pm)}M$. Now, we construct the symplectic paths

$$\Psi^\pm(t) = \Phi^\pm(t)^{-1} d\psi_t(x_\pm) \Phi^\pm(0).$$

It is easy to see that $\Psi^\pm(1) \in Sp^*(2n)$ since x_\pm are nondegenerate. Then we can define the Conley-Zehnder index of x_\pm by

$$\mu_{CZ}(x_\pm) = \mu_{CZ}(\Psi^\pm),$$

and the Maslov index of the pair (u, H) is defined by

$$\mu(u, H) = \mu_{CZ}(x_+) - \mu_{CZ}(x_-) = \mu_{CZ}(\Psi^+) - \mu_{CZ}(\Psi^-),$$

which is independent of the choice of the trivialization and satisfies the following properties

(*Homotopy*): For given H and two nondegenerate fixed points x_{\pm} of ϕ_H , $\mu(u, H)$ is constant on the homotopy components of $\mathcal{P}(x_-, x_+)$.

(*Zero*): If $x_- = x_+$, then $\mu(u, H) = 0$.

(*Catenation*): $\mu(u_{01} \# u_{12}, H) = \mu(u_{01}, H) + \mu(u_{12}, H)$.

(*Chern class*): For $v : S^2 \rightarrow M$, $\mu(u \# v, H) = \mu(u, H) - 2c_1(v)$.

(*Morse index*): If $\phi = Id$ and $H_t = H$ is a Morse function with sufficiently small second derivatives, then fixed points are the critical points of H and $\mu(u, H) = Ind_H(x_+) - Ind_H(x_-)$.

(*Fixed point index*): For $u \in \mathcal{P}(x_-, x_+)$,

$$(-1)^{\mu(u, H)} = \text{sign of } \det(Id - d\phi(x_+)) \det(Id - d\phi(x_-)).$$

Let $\widetilde{\text{Fix}}(\phi_H) \subset \widetilde{\Omega}_\phi$ denote the set of elements which cover curves of the form $x(t) = \psi_t(x)$, $x \in \text{Fix}(\phi_H)$. Then we have a fibration of discrete sets

$$\Gamma \hookrightarrow \widetilde{\text{Fix}}(\phi_H) \rightarrow \text{Fix}(\phi_H).$$

Every function $u \in \mathcal{P}(x_-, x_+)$ and every lift $\tilde{x}_- \in \widetilde{\text{Fix}}(\phi_H)$ of x_- determines a unique lift $\tilde{x}_+ = \tilde{x}_- \# u$ of x_+ . By the “homotopy” and “catenation” properties of the Maslov index there exists a unique map $\mu_{rel} : \widetilde{\text{Fix}}(\phi_H) \times \widetilde{\text{Fix}}(\phi_H) \rightarrow \mathbb{Z}$ such that

$$\mu(u, H) = \mu_{rel}(\tilde{x}_-, \tilde{x}_+)$$

whenever $\tilde{x}_+ = \tilde{x}_- \# u$. Then by the “Chern class” property one has

$$\mu_{rel}(\tilde{x}_-, v \# \tilde{x}_+) = \mu_{rel}(\tilde{x}_-, \tilde{x}_+) - 2c_1(v)$$

for $v \in \pi_2(M)$, and the “catenation” property reads

$$\mu_{rel}(\tilde{x}_0, \tilde{x}_1) + \mu_{rel}(\tilde{x}_1, \tilde{x}_2) = \mu_{rel}(\tilde{x}_0, \tilde{x}_2).$$

Now, if we fix a reference critical point of the functional F , say $\tilde{x}_0 \in \widetilde{\text{Fix}}(\phi_H)$, such that its projection $x_0 \in \text{Fix}(\phi_H)$ is nondegenerate, then we can define the Conley-Zehnder index of any $\tilde{x} \in \text{Crit}(F)$ by

$$\mu_{CZ}(\tilde{x}) = \mu_{rel}(\tilde{x}_0, \tilde{x}).$$

It is easy to see that the dimension of the nonparameterized moduli space $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ is $\mu_{rel}(\tilde{x}_-, \tilde{x}_+) - 1$.

3 Moduli space of stable connecting orbits.

Now we follow the method by Fukaya-Ono[FO] with some modifications to define the stable connecting orbits and moduli space. We denote the set of critical points of the function F by $\text{Crit}(F)$. For convenience of the reader, we list some necessary definitions.

Definition 3.1 A semi-stable curve with k marked points is a pair (Σ, \mathbf{z}) , where the set $\Sigma = \cup \pi_{\Sigma_\nu}(\Sigma_\nu)$ is connected, each Σ_ν is a Riemann surface, and the number of Σ_ν is finite, $\pi_{\Sigma_\nu} : \Sigma_\nu \rightarrow \Sigma$ is a continuous and locally homeomorphic map, and $\mathbf{z} = (z_1, \dots, z_k)$ are k distinguished points on Σ . Moreover, the following hold

- 1° For each $p \in \Sigma$, $\text{Sum}_\nu \# \pi_{\Sigma_\nu}^{-1}(p) \leq 2$; For each z_i , $\text{Sum}_\nu \# \pi_{\Sigma_\nu}^{-1}(z_i) = 1$.
- 2° The set $\{p | \text{Sum}_\nu \# \pi_{\Sigma_\nu}^{-1}(p) = 2\}$ is of finite order.

If $\text{Sum}_\nu \# \pi_{\Sigma_\nu}^{-1}(\pi_{\Sigma_\nu}(p)) = 2$, p is called a *singular or double point*. If $\pi_{\Sigma_\nu}(p) = z_i$ for some i , we say the $p \in \Sigma_\nu$ is *marked*. And say Σ_ν is a *component* of Σ . If all components are spheres, we say (Σ, \mathbf{z}) is a genus 0 semi-stable curve.

A homeomorphism $h : (\Sigma, \mathbf{z}) \rightarrow (\Sigma, \mathbf{z})$ is called an *automorphism* if it can be lifted to bi-holomorphic isomorphisms $h_{\mu\nu} : \Sigma_\mu \rightarrow \Sigma_\nu$ for each component and $h(z_i) = z_i$ for each i . We denote the automorphism group of (Σ, \mathbf{z}) by $\text{Aut}(\Sigma, \mathbf{z})$ or G_Σ .

Definition 3.2 Let (M, ω) be a symplectic manifold with a compatible almost complex structure $J : TM \rightarrow TM$. A continuous map $u : \Sigma \rightarrow M$ is called *pseudo-holomorphic* if the composition $u \circ \pi_{\Sigma_\nu} : \Sigma_\nu \rightarrow M$ is *pseudo-holomorphic* (or *J-holomorphic*) for each ν .

Definition 3.3 A pair $((\Sigma, \mathbf{z}), u)$ is called a *J-stable map* if for each ν one of the following conditions holds

- 1° $u \circ \pi_{\Sigma_\nu} : \Sigma_\nu \rightarrow M$ is a nonconstant *J-holomorphic map*.
- 2° Let m_ν be the number of special points on Σ_ν which are singular or marked, then $m_\nu \geq 3$.

If a semi-stable curve (Σ, \mathbf{z}) satisfies the condition 2° in the Definition 3.3 for each component, then we say it is a *stable curve*. For a pair $((\Sigma, \mathbf{z}), u)$, we define its *automorphism group* by

$$\text{Aut}((\Sigma, \mathbf{z}), u) = \{h : \Sigma \rightarrow \Sigma | h \text{ is an automorphism, and } u \circ h = u\}.$$

It can be proved that $((\Sigma, \mathbf{z}), u)$ is stable if and only if $\text{Aut}((\Sigma, \mathbf{z}), u)$ is a finite group[FO].

Definition 3.4 Let $\beta \in H_2(M, \mathbb{Z})$. We denote by $(M, J, \beta)_{0,k}$ for the set of genus 0 stable maps $((\Sigma, \mathbf{z}), u)$ with k marked points such that $u_*([\Sigma]) = \beta$. We say two stable maps are equivalent, i.e. $((\Sigma, \mathbf{z}), u) \sim ((\Sigma, \mathbf{z}'), u')$, if and only if there exists an isomorphism $h : (\Sigma, \mathbf{z}) \rightarrow (\Sigma, \mathbf{z}')$ satisfying $u' \circ h = u$ and $h(z_i) = z'_i$ for each i . We write $\text{CM}_{0,k}(M, J, \beta) = (M, J, \beta)_{0,k} / \sim$, and call it the *moduli space of this kind of stable maps*.

We define the energy of a genus 0 stable map $((\Sigma, \mathbf{z}), u) \in (M, J, \beta)_{0,k}$ by

$$E(u) = E((\Sigma, \mathbf{z}), u) = \int_{\Sigma} u^* \omega$$

Recall that $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ is the moduli space of gradient flow lines connecting two critical points \tilde{x}_-, \tilde{x}_+ of F . Note the periodicity condition (2), we give the following

Definition 3.5 *A stable connecting orbit between $\tilde{x}_- = \tilde{x}_1$ and $\tilde{x}_+ = \tilde{x}_{K+1}$ is a triple $((v_1, \dots, v_K), (f_1, \dots, f_l), o)$ such that*

(1) $v_j = u_j|_{\mathbb{R} \times [0,1]}$, where $u_j = \pi(\tilde{u}_j)$, $\tilde{u}_j \in \mathcal{M}(\tilde{x}_j, \tilde{x}_{j+1})$, $\tilde{x}_j \in \text{Crit}(F)$, $j = 1, \dots, K+1$.

(2) $f_i = (\Sigma_{f_i}, u_{f_i}) \in \mathcal{CM}_{0,1}(M, J, \beta_i)$, where Σ_{f_i} is a genus zero semi-stable curve with one marked point and $u_{f_i} : \Sigma_{f_i} \rightarrow M$, and $[u_{f_i}(\Sigma_{f_i})] = \beta_i$. Let $z_i \in \Sigma_{f_i}$ be the marked point.

(3) o is an injection from $\{1, \dots, l\}$ to the K copies of $\mathbb{R} \times [0, 1]$. If $o(i) = (s_i, t_i)$ is on the j^{th} copy of $\mathbb{R} \times [0, 1]$, we require that $u_{f_i}(z_i) = v_j(s_i, t_i)$. Moreover, if there exists some $i \in \{1, \dots, l\}$ satisfying $o(i) = (s_i, 0)$ or $o(i) = (s_i, 1)$, then there exists a $j \in \{1, \dots, l\}$ such that $o(j) = (s_j = s_i, 1)$ or $o(j) = (s_j = s_i, 0)$, respectively.

(4) If $\tilde{x}_j = \tilde{x}_{j+1}$, then there exists an i such that $o(i)$ is on the j^{th} copy of $\mathbb{R} \times [0, 1]$.

We say each v_j is a *main component* and each component of f_i is a *bubble component* of the stable connecting orbit. In particular, sometimes if necessary, we denote by f_i^B for the distinct f_i above satisfying $o(i) = (s_i, 0)$ or $o(i) = (s_i, 1)$ and by f_i^I for others.

Since each v_j is the restriction of u_j , for a sequences $\{v_j^{(n)}\}$, if there exists a v_j^* such that $v_j^{(n)} \rightarrow v_j^*$ if and only if there exists a map u_j^* such that $v_j^* = u_j^*|_{\mathbb{R} \times [0,1]}$ and $u_j^{(n)} \rightarrow u_j^*$.

Naturally, we define the *domain* Σ of the stable connecting orbit as K copies of $\mathbb{R} \times [0, 1]$ with two particular segments of line $L_{\pm\infty} = \{\pm\infty\} \times [0, 1]$, which are called the *main components*, denoted by $\Sigma_{m,i}$, $i = 1, \dots, K$, attached with l genus zero semi-stable curves $\Sigma_{f_1}, \dots, \Sigma_{f_l}$ at l points $o(1), \dots, o(l)$, whose components are called the *bubble components* which each one can be identified with S^2 , we assume there totally are L such bubble components, denoted them by $\Sigma_{b,j}$, $j = 1, \dots, L$. And the double points include all $o(i)$ and the singular points on all Σ_{f_i} which are the intersections of bubble components in Σ_{f_i} . To make the domain be stable curve, we can add some marked points on those semi-stable bubble components such that the condition 2° in Definition 3.3 holds, also in order to consider the bubbling-off process we maybe add some markings on main components as well. We call such curve is \mathcal{P} -stable.

We can simply write the stable connecting orbit as (Σ, V) , where $V : \Sigma \rightarrow M$ satisfying $V = v_j$ on the j^{th} copy of $\mathbb{R} \times [0, 1]$, and $V = u_{f_i}$ on Σ_{f_i} .

For $\mathbf{r} = (r_1, \dots, r_K) \in \mathbb{R}^K$, it is easy to see that there exists a natural \mathbb{R}^K -action on (Σ, V) . If $o(i)$ is on the j^{th} component, also the action is to translate $o(i)$ by

$r_j \in \mathbb{R}$. We say $\mathbf{r}(\Sigma, V)$ is equivalent to (Σ, V) , and denote $[\Sigma, V]$ for the equivalent class of (Σ, V) .

Now we define the moduli space of stable connecting orbits $P\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$,

Definition 3.6 *we say $[\Sigma, V] \in P\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ if $\tilde{x}_{K+1} = \tilde{x}_+$ and*

$$\tilde{x}_1 \# (\beta_1 + \cdots + \beta_l) = \tilde{x}_-.$$

We define the energy of $[\Sigma, V] \in P\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ is

$$E([\Sigma, V]) = \sum_{j=1}^K E(u_j) + \sum_{i=1}^{l_1} E(f_i^I) + \frac{1}{2} \sum_{i=1}^{l_2} E(f_i^B), \quad (5)$$

where $l_1 = \#\{f_i^I\}$, $l_2 = \#\{f_i^B\}$, and $l_1 + l_2 = l$.

We can see that the energy only depends on \tilde{x}_-, \tilde{x}_+ , i.e. $E([\Sigma, V]) = F(\tilde{x}_+) - F(\tilde{x}_-)$, and is independent of the choice of $[(v_1, \cdots, v_K), (f_1, \cdots, f_l), o] \in P\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$.

The following lemma shows that the energy of each genus 0 stable map or each nonconstant connecting orbit is bounded from below by a constant.

Lemma 3.1 *Given a compact symplectic manifold (M, ω) with compatible almost complex structure J . There exists a constant $\delta > 0$ such that if $\tilde{x}_- \neq \tilde{x}_+$, then for any nonconstant genus zero pseudo-holomorphic map $f : \Sigma \rightarrow M$ and $\forall u \in \mathcal{M}(\tilde{x}_-, \tilde{x}_+)$, $\min(E(f), E(u)) > \delta$.*

The proof is standard, we refer the reader to [G][FO].

In order to construct a Fredholm system defined in the Definition 1.1, we define an ambient space of $P\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$. Firstly, we introduce the notion of stable $W^{k,p}$ or $L^{k,p}$ -connecting orbits with $k - \frac{2}{p} > 1$. Given a set Σ which is the domain defined above.

Definition 3.7 *We say a triple $((v_1, \cdots, v_K), (f_1, \cdots, f_l), o)$ is a stable $W^{k,p}$ -connecting orbit between $\tilde{x}_- = \tilde{x}_1$ and $\tilde{x}_+ = \tilde{x}_{K+1}$ if*

1° $v_j = u_j|_{\mathbb{R} \times [0,1]}$, where $u_j = \pi(\tilde{u}_j)$ is a $W^{k,p}$ -map such that $\lim_{s \rightarrow -\infty} \tilde{u}_j(s, \cdot) = \tilde{x}_j$, $\lim_{s \rightarrow +\infty} \tilde{u}_j(s, \cdot) = \tilde{x}_{j+1}$, $\tilde{x}_j \in \text{Crit}(F)$, $j = 1, \cdots, K+1$.

2° $f_i = (\Sigma_{f_i}, u_{f_i})$, each $u_{f_i} : \Sigma_{f_i} \rightarrow M$ is a stable $W^{k,p}$ -map, and $[u_{f_i}(\Sigma_{f_i})] = \beta_i$.

3° The conditions (3) and (4) in the Definition 3.5 are satisfied.

4° $\tilde{x}_{K+1} = \tilde{x}_+$ and $\tilde{x}_1 \# (\beta_1 + \cdots + \beta_l) = \tilde{x}_-$.

5° Each $W^{k,p}$ -map u_j satisfies the following exponential $W^{k,p}$ -decay condition along its ends $x_j = \pi \tilde{x}_j$ and $x_{j+1} = \pi(\tilde{x}_{j+1})$

$$\int \int_{\mathbb{R} \times [0,1]} (|\eta_{j-1}^{(m)}|^p + |\eta_j^{(m)}|^p) e^{\varepsilon_0 |s|} ds dt < \infty, \quad m = 0, 1, \cdots, k,$$

where η_j is defined by $u_j(s, t) = \exp_{x_j} \eta_j(s, t)$ for sufficiently large s , and ε_0 is a fixed small positive constant.

We denote the moduli space of equivalence classes of all such stable $W^{k,p}$ -orbits connecting \tilde{x}_- and \tilde{x}_+ by $\mathcal{B}(\tilde{x}_-, \tilde{x}_+)$. All other notation introduced for stable connecting orbits above are also applicable to stable $W^{k,p}$ -orbits. Sometimes, we will simply write $P\mathcal{M}$ and \mathcal{B} for these spaces defined above with the ends \tilde{x}_\pm being not expressed explicitly, and write a stable orbit, i.e. the triple $((v_1, \cdots, v_K), (f_1, \cdots, f_l), o)$ in $P\mathcal{M}$ and \mathcal{B} by (Σ, V) or V .

4 Gluing.

In this section, we will define a small neighborhood \mathcal{W} of the stable moduli space $PM(\tilde{x}_-, \tilde{x}_+)$ in its ambient space $\mathcal{B}(\tilde{x}_-, \tilde{x}_+)$, and we can see on \mathcal{W} there exists a naturally defined topology which is Hausdorff. The moduli space PM is a compact subset of \mathcal{W} . And in some suitably abstract settings, we can consider \mathcal{W} as a space with two topologies, say a *partially smooth* space. On this space one can define a so-called *multi-fold* atlas, and there are related *multi-bundles* over it and a family of compatible multi-valued sections, say *multi-section*, which can be used to define the virtual cycle. The application of the settings in this section will give rise to the construction of virtual moduli cycle and the definition of Floer homology in the section 6.

In order to study the moduli space PM , we have to give a description of the domain. Indeed, we will see that there exist stratifications $PM = \cup_D PM^D$ and $\mathcal{W} = \cup_D \mathcal{W}^D$. The problem is to describe how the strata \mathcal{W}^D fit together as the topological type of the domain changes. The gluing method can give the local “cornered” coordinate chart (or say uniformizer) of the space of domains.

4.1 Gluing the domains

Firstly, we give a short description of the domains of stable connecting orbits and the structure of moduli space $\overline{PM}_{0,k}$ of such open stable curves, which we call \mathcal{P} -stable curves. Following the notion above, we denote such a k -pointed \mathcal{P} -stable curve by $(\Sigma, \mathbf{z}) = (\Sigma, z_1, \dots, z_k)$. Recall that Σ is the union of K main components $\Sigma_{m,i}$, $i = 1, \dots, K$, which are copies of $\mathbb{R} \times [0, 1]$ and L bubble components $\Sigma_{b,j}$, $j = 1, \dots, L$, which are identified with sphere. Such a stable curve is said to be \mathcal{P} -stable if the following things hold. Each $\Sigma_{m,i}$ has two particular segments of line $L_{i,\pm\infty} \simeq \{\pm\infty\} \times [0, 1]$. All main components together form a chain such that $L_{i,+\infty} = L_{i+1,-\infty}$, $i = 1, \dots, K-1$. Sometimes we only write Σ_m and Σ_b for a main and a bubble component if without danger of confusion.

We say a main component Σ_m of Σ is free if it has no double point, and say a bubble component Σ_b of Σ is free if it has at most two double points. In order to get a stable curve Σ^s with minimal marked points, we may at first use the “forgetting marking” procedure to send $(\Sigma, z_1, \dots, z_k)$ to Σ^u by ignoring all marked points $z_i, i = 1, \dots, k$, then add one or two markings on each free bubble component of Σ^u to make it stable.

We denote by G_m, G_b the automorphism group of Σ_m and Σ_b , respectively. Note that on the free main component, G_m is the group of all \mathbb{R} -translations acting on Σ_m , and on the free bubble component, G_b is the holomorphic spherical automorphism group preserving double points of Σ_b . Note that the automorphism group G_Σ of Σ consists of all holomorphic isomorphisms of Σ after forgetting its marked points, and it may interchange different components of Σ , we call it the reparameterization group. It contains G_m and G_b as subgroups.

Two \mathcal{P} -stable curves Σ_1 and Σ_2 are called to be equivalent if there is a homeomorphism $\varphi : \Sigma_1 \rightarrow \Sigma_2$ preserving the marked points and the boundary lines of all main components such that the restriction of φ to each component of Σ_1 is a

holomorphic map. We denote by $\mathcal{P}M_{0,k}$ the collection of all equivalence class of k -pointed \mathcal{P} -stable curves with only one main component and no bubble component. Its stable compactification is denoted by $\overline{\mathcal{P}M}_{0,k}$, which is just all \mathcal{P} -stable curves defined above. Roughly speaking, we can use the splitting process of main components and let some of marked points go together or go to the boundary of $\mathbb{R} \times [0, 1]$ symmetrically to obtain $\overline{\mathcal{P}M}_{0,k}$ from $\mathcal{P}M_{0,k}$.

The topological type of the curve Σ is determined by its number of main components and intersection pattern, say I_Σ , which pairwise corresponds to the specific lines $L_{i,+\infty} = L_{i+1,-\infty}$ in main components and points in the smooth resolution $\tilde{\Sigma}$ of Σ that correspond just those double points in Σ . There are of course finite many main components and intersection patterns. We use the notation

$$\mathcal{I} = \{I_\Sigma | \Sigma \in \overline{\mathcal{P}M}_{0,k}\}$$

to describe the collection of all topological types.

$\overline{\mathcal{P}M}_{0,k}$ is stratified according to the topological type I_Σ of a curve Σ in a stratum. We can write

$$\overline{\mathcal{P}M}_{0,k} = \cup_{I \in \mathcal{I}} \mathcal{P}M_{0,k}^I,$$

where $\mathcal{P}M_{0,k}^I$ is the collection of curves with fixed intersection pattern $I_\Sigma = I$. Each $\mathcal{P}M_{0,k}^I$ is a smooth manifold.

The element $\Sigma \in \overline{\mathcal{P}M}_{0,k}$ will naturally appear as domains of the stable (J, H) -connecting orbits. We can always start from the case Σ^s which is Σ equipped with minimal number of marked points needed for stability. In the bubbling process, the topological type of domains maybe change, then we will also consider those stable curves with extra markings.

Now for a fixed $I = I_{\Sigma^s}$ and a fixed element $\Sigma \in \mathcal{P}M_{0,k}^I$, we follow the idea of Liu-Tian to give a local description of the nearby curves in $\overline{\mathcal{P}M}_{0,k}$.

Since $\Sigma = \Sigma^s$, there is no extra marked point. That is to say, there are at most two marked points on each component Σ_l . We denote the double points by $d_{l,k}$. So the locations of all double points for the nearby Σ' can be regarded as a local coordinate of $\mathcal{P}M_{0,k}^I$ near Σ . If the double point $d_{l,k}$ lies in the inner part of the component Σ_l , we just let $\alpha_{l,k} \in D_\delta(d_{l,k})$ be the complex coordinate of the δ -disc centering at $d_{l,k}$. If otherwise $d_{l,k}$ lies in the boundary of a main component $\Sigma_l \simeq \mathbb{R} \times [0, 1]$ (certainly $d_{l,k} = d_{l',k'}$ maybe simultaneously lie in the bubble component $\Sigma_{l'}$), then let $\alpha_{l,k}$ be the complex coordinate of the closed half δ -disc centering at $d_{l,k}$, denoted by $HD_\delta(d_{l,k})$. From the φ -periodicity condition (2) for the solution of perturbed Cauchy-Riemann equation (1) we know that if bubbling-off occur in the boundary of a main component Σ_l , they should appear synchronously on the two sides of $\Sigma_l \simeq \mathbb{R} \times [0, 1]$ satisfying also a φ -periodicity. So if $d_{l,k} = (s, \{0\})$ naturally we have the other double point $d_{l,k'} = (s, \{1\})$ with complex coordinate $\alpha_{l,k'}$ in the other closed half δ' -disc centering at $d_{l,k'}$. The collection $\alpha = (\alpha_{l,k})$ is the local coordinate of $\mathcal{P}M_{0,k}^I$ near Σ . The corresponding curve is denoted by Σ_α .

Then for each double point on the nearby curve $\Sigma' = \Sigma_\alpha$, corresponding to two intersecting components Σ'_{l_1} and Σ'_{l_2} , we have $d'_{l_1,k_1} = d'_{l_2,k_2}$. If one double point, say d'_{l_1,k_1} is on the boundary of a main component, we can just temporarily consider a little larger domain containing Σ'_{l_1} , i.e. with a larger component $\bar{\Sigma}'_{l_1} \simeq$

$\mathbb{R} \times [0 - \epsilon, 1 + \epsilon]^2$. Then we can set a complex gluing parameter $t_{l_1, k_1} = t_{l_2, k_2}$ in the δ -disc centering at the origin of \mathbb{C} , and for each pair of lines $L_{i, +\infty} = L_{i+1, -\infty}$ in two connecting main components Σ'_i and Σ'_{i+1} , a real gluing parameter $\tau_i \in [0, \delta]$. Denote the collection of all parameters by $(\alpha, t, \tau) = \{(\alpha_{l, k}, t_{l, k}, \tau_i)\}$. Then the following procedure shows how to get a curve $\Sigma_{(\alpha, t, \tau)}$ with different topological type from the curve $\Sigma' = \Sigma_\alpha$.

For each double point $d'_{l_1, k_1} = d'_{l_2, k_2}$ of Σ_α with coordinates α_{l_1, k_1} and α_{l_2, k_2} , take complex coordinates z_{l_1, k_1} and z_{l_2, k_2} in the two discs $D_{\delta'}(\alpha_{l_1, k_1}) \subset \Sigma'_{l_1}$ (or $\bar{\Sigma}'_{l_1}$) and $D_{\delta'}(\alpha_{l_2, k_2}) \subset \Sigma'_{l_2}$ respectively. Suppose $z = e^{-2\pi(r+i\theta)}$, then (r, θ) is the corresponding cylindrical coordinate. We firstly cut off discs

$$\{(r_{l_1, k_1}, \theta_{l_1, k_1}) \mid r_{l_1, k_1} > -\log|t_{l_1, k_1}|\} \text{ in } D_{\delta'}(\alpha_{l_1, k_1})$$

and

$$\{(r_{l_2, k_2}, \theta_{l_2, k_2}) \mid r_{l_2, k_2} > -\log|t_{l_2, k_2}|\} \text{ in } D_{\delta'}(\alpha_{l_2, k_2}),$$

then along their boundaries and according to the formula

$$\theta_{l_1, k_1} = \theta_{l_2, k_2} + \arg(t_{l_1, k_1} = t_{l_2, k_2}),$$

we glue back the remaining parts of $D_{\delta'}(\alpha_{l_1, k_1})$ and $D_{\delta'}(\alpha_{l_2, k_2})$. If d'_{l_1, k_1} is on the boundary of the main component Σ'_{l_1, k_1} , after above gluing procedure we only get a mid-step curve $\bar{\Sigma}_{(\alpha, t)}$ with some new larger main component $\bar{\Sigma}_{(\alpha, t), l_1}$, one more thing we should do is to cut off the additional margin of $\bar{\Sigma}_{(\alpha, t), l_1}$ to obtain a suitable main component $\Sigma_{(\alpha, t), l_1}$.

To use the real parameter τ to glue two connecting main components Σ'_i and Σ'_{i+1} of Σ_α along the line $L_{i, +\infty} = L_{i+1, -\infty}$ is much simpler and direct. Let $\tau_i^+ = \tau_{i+1}^-$, we cut off a strip $[\frac{1}{\tau_i^+}, +\infty) \times [0, 1]$ in the main component Σ'_i and a strip $(-\infty, \frac{1}{\tau_{i+1}^-}] \times [0, 1]$ in Σ'_{i+1} , then we just simply glue the two remaining parts with identifying the two end lines.

So the resulting curve is just $\Sigma_{(\alpha, t, \tau)}$, which is an element of $\overline{\mathcal{PM}}_{0, k}$ near Σ . The parameter (α, t, τ) is a ‘‘cornered’’ coordinate chart of $\overline{\mathcal{PM}}_{0, k}$ near Σ .

We can see that there is an obvious partial order for the collection of various topological types, *i.e.* $I_1 > I$ if the topological type of Σ_{I_1} can be obtained from Σ_I by above gluing procedure. Moreover, $\Sigma_\alpha \in \mathcal{PM}_{0, k}^I$ if and only if $t = 0$ and $\tau = 0$. Actually, we can get various curves in $\mathcal{PM}_{0, k}^{I_1}$ with $I_1 > I$ by setting some of components of (t, τ) be zero.

- *Compactness of stable moduli space*

We can define the Gromov-Floer topology, or for simplicity, say weak topology on the moduli space $\mathcal{PM}(\tilde{x}_-, \tilde{x}_+)$,

Definition 4.1 A sequence $[\Sigma_n, v_n] = [(v_1, \dots, v_K), (f_1, \dots, f_l), o]$ or $[V_n]$, $n = 1, \dots, \infty$, of stable connecting orbits is said to Gromov-Floer weakly converge to

²We have stated that such double points will appear simultaneously in pair at the two sides of the boundary, so the following gluing operating will be done simultaneously for the other double point near the other side of boundary of Σ'_{l_1} .

a stable connecting orbit $[\Sigma, v] = [(v_1, \dots, v_K), (f_1, \dots, f_l), o]_\infty$ or $[V_\infty]$ if there exist representatives $V_n \in [V_n]$ and $V_\infty \in [V_\infty]$ with domains Σ_n and Σ_∞ such that

1° $\Sigma_n \rightarrow \Sigma_\infty$, as $n \rightarrow \infty$. This means that $\exists \Sigma'_n \in \mathcal{PM}_{0,k}^{I_{\Sigma_n}}$, $\Sigma'_\infty \in \mathcal{PM}_{0,k}^{I_{\Sigma_\infty}}$ with minimal marked points and identification maps ϕ_n, ϕ_∞ satisfying $\Sigma_n = \phi_n(\Sigma'_n)$, $\Sigma_\infty = \phi_\infty(\Sigma'_\infty)$ such that when n is sufficiently large Σ'_n is in the neighborhood of Σ'_∞ and is represented by $\Sigma'_{(\alpha_n, t_n, \tau_n)}$ with $(\alpha_n, t_n, \tau_n) \rightarrow 0$.

2° For each compact set $K \subset \Sigma'_\infty \setminus \{\text{double points}\} \cup \{\cup_i L_{i, \pm\infty}\}$, if n is sufficiently large, let K_n be the corresponding subset of Σ'_n via gluing in $\mathcal{PM}_{0,k}$, then $(V_n \circ \phi_n)|_K$ is C^∞ -convergent to $(V_\infty \circ \phi_\infty)|_K$.

3° $\lim_{n \rightarrow \infty} E(V_n) = E(V_\infty)$.

Using the same method as in [LT1] (or in [RT] [FO], etc.), we can prove

Theorem 2 $\mathcal{PM}(\tilde{x}_-, \tilde{x}_+)$ is Hausdorff and compact in the sense of weak topology. Moreover, if $\{[V_n]\}_{n=1}^\infty \rightarrow [V]_\infty$ in $\mathcal{PM}(\tilde{x}_-, \tilde{x}_+)$, then $E(V_n) \rightarrow E(V_\infty)$ and for sufficiently large n , $\mu(V_n, H) = \mu(V_\infty, H)$.

Remark. Recall the Definition 3.5 we see that each main component v_i is the restriction of $u_i = \pi(\tilde{u}_i)$, where $\tilde{u}_i \in \mathcal{M}(\tilde{x}_i, \tilde{x}_{i+1})$ is the lift of a (J, H) -holomorphic map from the open domain \mathbb{R}^2 , to the strip $\mathbb{R} \times [0, 1]$. The Gromov weak compactness arguments of Floer in [F3] are carried out for the special case $\phi = id$, and can be easily generalized to arbitrary ϕ to prove the compactness of the moduli space $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ modulo splitting. Then for our moduli space of stable connecting orbits $\mathcal{PM}(\tilde{x}_-, \tilde{x}_+)$, we just need do the same bubbling-off analysis as [LT1] or [FO] restricting on the subset $\mathbb{R} \times [0, 1]$. Also we require that the bubble components appear simultaneously on both sides of the boundary because of the ϕ -periodicity condition (2). Here we will not repeat the proof which can be found in the references listed above.

So the moduli space $\mathcal{PM}(\tilde{x}_-, \tilde{x}_+)$ is the stable compactification of the moduli space of connecting orbits $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$.

4.2 Small neighborhood of stable moduli space

Following [LT1], with the difference in that our main components are not cylinders but strips, we give a sketchy description of the deformation of stable orbits under the topological change of their domains. We firstly consider stable orbits with fixed intersection pattern, then use the gluing procedure to deal with the stable orbits with different intersection patterns. Then we get a neighborhood \mathcal{W} of \mathcal{PM} , which can be locally uniformized and is a (partially smooth) orbifold.

Each stable orbit (Σ, V) or V consists of some main components, denoted by V_m , and some bubble components, denoted by V_b . Each V determines an intersection pattern D_V which is determined by (I) the intersection pattern I_Σ of the domain Σ and (II) the relative homotopy class of each main component V_m fixing its two end lines $L_{m, \pm\infty}$ and the homology class represented by each bubble component V_b . Recall the definition 3.7 we see that each $V \in \mathcal{B}(\tilde{x}_-, \tilde{x}_+)$ has the so-called *effective* intersection pattern D_V defined by Liu-Tian[LT1]. In general, in the $W^{k,p}$ -category,

we say that an intersection pattern D is effective if $D = D_V$ with V being a stable (J, H) -map. So we can define the energy of each V or D by

$$E(V) = E(D = D_{V'}) = E(V'), \quad V' \in P\mathcal{M},$$

where $E(V')$ is defined as (5). If we denote the set of intersection pattern with bounded energy by $D^e = \{D | E(D) \leq e\}$, then using the Lemma 3.1 we know that there are at most finite number of marked points and ghost bubble components, consequently, the set D^e is finite.

Thus, according to intersection pattern, we can stratify $P\mathcal{M} = \cup_D P\mathcal{M}^D$, where each

$$P\mathcal{M}^D(\tilde{x}_-, \tilde{x}_+) = \{[V] | V \in P\mathcal{M}(\tilde{x}_-, \tilde{x}_+), D_V = D\},$$

and similarly, denote the stratified set of $W^{k,p}$ -orbits with bounded energy $E(V) \leq e$ by $\mathcal{B}^e = \cup_D \mathcal{B}^{e,D}$. Since for Given \tilde{x}_\pm , the energy $E(V)$ of any stable connecting orbit $[V] \in P\mathcal{M}$ is bounded, if e is sufficiently large, then $P\mathcal{M}(\tilde{x}_-, \tilde{x}_+) \subset \mathcal{B}^e(\tilde{x}_-, \tilde{x}_+)$. We will use \mathcal{B}^e as the ambient space of $P\mathcal{M}$, after taking an e once for all, we will omit the superscript and still denote the space by \mathcal{B} . We firstly study the space \mathcal{B}^D with fixed intersection pattern D and therefore I_Σ of their domains.

Since the reparameterization group G_Σ is noncompact, we have no nice structure of \mathcal{B}^D . While, we will show that near $P\mathcal{M}^D$ the action G_Σ has a good slicing. To this end, we choose a representative V of $[V] \in P\mathcal{M}^D$. Let V_m , $m = 1, \dots, M$ and V_b , $b = 1, \dots, B$ be its free main and bubble components. For simplicity, we assume each main or bubble free component has only one generic marked point, say $(0, \frac{1}{2})$ and 0, respectively. We take locally a codimension 1 small disc \mathbf{H}_m near $V_m(0, \frac{1}{2})$ for each free main component V_m such that $V_m|_{\mathbb{R} \times \{\frac{1}{2}\}}$ is transversal to \mathbf{H}_m , and a codimension 2 small disc \mathbf{H}_b near $V_b(0)$ for each free bubble component V_b such that V_b is transversal to \mathbf{H}_b at 0. Let

$$\mathbf{H} = \prod_{m=1}^M \mathbf{H}_m \times \prod_{b=1}^B \mathbf{H}_b.$$

We define the distance in \mathcal{B}^D (each element has $K+L$ components and d double points) as $\|V - V'\|_{\mathcal{B}^D} = \sum_{i=1}^{K+L} \|V_i - V'_i\|_{k,p} + \sum_{j=1}^d \text{dist}(z_j - z'_j)$, where dist is the distance function in the domain Σ and the Sobolev number k, p will be taken carefully so that the $W^{k,p}$ -norm should be stronger than the C^1 -topology. Now for a sufficiently small ϵ -neighborhood $\tilde{U}_\epsilon^D(V) = \{W | \|W - V\|_{\mathcal{B}^D} \leq \epsilon\}$ (where the ϵ is needed small enough so that $[W(\Sigma_j)] = [V(\Sigma_j)]$ for all j), we can define a slicing (at least with respect to those group actions of $\prod_{m=1}^P G_m$) of $\tilde{U}_\epsilon^D(V)$ as

$$\tilde{U}_\epsilon^D(V, \mathbf{H}) = \{W | W \in \tilde{U}_\epsilon^D(V), W_m(0, \frac{1}{2}) \in \mathbf{H}_m, W_b(0) \in \mathbf{H}_b\},$$

with taking m from 1 to M and b from 1 to B . So the problem is to deal with the bubble component V_b .

Recall G_Σ is the reparameterization group of stable maps. We denote the automorphism group of V by $\Gamma_V = \{g | g \in G_\Sigma, V \circ g = V\}$. Γ_V is a finite group since V is stable. And it is generated by the subgroup $\prod_{m=1}^K \Gamma_{V_m} \times \prod_{b=1}^L \Gamma_{V_b} \times \Gamma_I$, where

$$\Gamma_{V_{m,b}} = \{g_{m,b} | V_{m,b} \circ g_{m,b} = V_{m,b}, \text{preserving double points}\}$$

and

$$\Gamma_I = \{g \in G_\Sigma \mid \text{interchanging components of } \Sigma \text{ and preserving } V\}.$$

If the automorphism group Γ_V is trivial, for small enough ϵ , the projection $\pi_V : \tilde{U}_\epsilon^D(V, \mathbf{H}) \rightarrow \mathcal{W}^D$ which takes the point W to its equivalence class $[W]$ is injective, thus $[V]$ has a neighborhood in \mathcal{W}^D modeled as an open subset in a Banach space. If Γ_V is nontrivial, that is to say, either some V_b is a multiple covering sphere or the automorphism interchanges components of Σ , then as in section 2 of [LT1], we can extend its action to a linear action on $\tilde{U}_\epsilon^D(V, \mathbf{H})$ in such a way that a neighborhood $U_{[V]}^D$ of $[V]$ in \mathcal{W}^D can be identified with the quotient $\tilde{U}_\epsilon^D(V, \mathbf{H})/\Gamma_V$. For simplicity, we only state how to extend the action of $\tilde{\Gamma}_V = \prod_{b=1}^L \Gamma_{V_b} = \Gamma_{V_b}$ (*i.e.* only one bubble component) to $\tilde{U}_\epsilon^D(V, \mathbf{H})$. The general case is in principle same. We denote the set of pre-image points of the bubble component W_b by

$$W_b^{-1}(W_b(0)) = \{y_1 = 0, y_2, \dots, y_{n_b}\}.$$

Similar to the Lemma 2.2 in [LT1] with the difference in that maybe some of y_i 's is on the boundary of the domain, we can obtain the analogous conclusion that for sufficiently small ϵ and δ , and for any $W \in \tilde{U}_\epsilon^D(V, \mathbf{H})$, there exist exactly n_b points, $y_1(W_b), \dots, y_{n_b}(W_b)$ such that for each i , $y_i(W_b)$ is in a δ -disc or half δ -disc centering at y_i (denoted by $D_\delta(y_i)$ or $HD_\delta(y_i)$), and

$$W_b^{-1}(\mathbf{H}_b) = \{y_1(W_b), \dots, y_{n_b}(W_b)\}.$$

Let g_i be the automorphism of S^2 such that $g_i(y_1) = y_i$, $i = 1, \dots, n_b$, $g_i(1) = 1$, $g_i(\infty) = \infty$, where we choose $y_1 = 0$. For any $W \in \tilde{U}_\epsilon^D(V, \mathbf{H})$, we define an automorphism of S^2 as

$$g_i^W : y_1 \rightarrow y_i(W), 1 \rightarrow 1, \infty \rightarrow \infty.$$

Let $r = \min_{i>m} \|V - V \circ g_i\|$. So if $\epsilon \ll \epsilon_1 \ll r$, then $W \circ g_i^W \in \tilde{U}_{\epsilon_1}^D(V, \mathbf{H})$ if and only if $i \leq m$. This gives rise to an action of $\tilde{\Gamma}_V$ on $\tilde{U}_{\epsilon_1}^D(V, \mathbf{H})$:

$$W * g = W \circ g^W,$$

for $W \in \tilde{U}_{\epsilon_1}^D(V, \mathbf{H})$, $g \in \tilde{\Gamma}_V$.

We also see that for any give two elements W_1 and W_2 in $W \in \tilde{U}_{\epsilon_1}^D(V, \mathbf{H})$, W_1 and W_2 are equivalent if and only if there exists a $g \in \Gamma_V$ such that $W_1 = W_2 * g$. So if we replace $\tilde{U}_{\epsilon_1}^D(V, \mathbf{H})$ by the Γ_V -invariant subset $\cup_{g \in \Gamma_V} g(\tilde{U}_\epsilon^D(V, \mathbf{H}))$, then the action constructed above is a smooth right action on $\cup_{g \in \Gamma_V} g(\tilde{U}_\epsilon^D(V, \mathbf{H}))$, and a neighborhood of $[V]$ in \mathcal{B}^D is homeomorphic to $\cup_{g \in \Gamma_V} g(\tilde{U}_\epsilon^D(V, \mathbf{H}))/\Gamma_V$. For simplicity of notation, we still write the Γ_V -invariant subset $\cup_{g \in \Gamma_V} g(\tilde{U}_\epsilon^D(V, \mathbf{H}))$ as $\tilde{U}_\epsilon^D(V, \mathbf{H})$ if no danger of confusion.

We say the triple $(\tilde{U}_\epsilon^D(V, \mathbf{H}), \Gamma_V, \pi_V)$ is a local uniformizer for $[V]$ in \mathcal{W}^D , where π_V is the quotient map. In other words, there exists a neighborhood \mathcal{W}^D of $P\mathcal{M}^D$ in the space \mathcal{B}^D of all stable orbits with intersection pattern D which is covered by local uniformizers $(\tilde{U}_\epsilon^D(V, \mathbf{H}), \Gamma_V, \pi_V)$. These uniformizers give the neighborhood an orbifold structure.

Now the problem is to describe how the strata \mathcal{W}^D fit together with topological change of the domain. We also need the gluing procedure to describe the local information of a stable orbit $V : \Sigma \rightarrow M$ in the full neighborhood \mathcal{W} . Assume the topological type of the domain Σ is $I = I_\Sigma$. Recall in the last subsection a δ -neighborhood of the element Σ in $\mathcal{PM}_{0,k}^{I_\Sigma}$ can be described by parameters $\alpha = (\alpha_{l,k})$ with $\alpha_{l,k} \in D_\delta(d_{l,k})$, and when the intersection pattern changes the full δ -neighborhood of Σ is described by (α, t, τ) with $\|t\|, |\tau| \leq \delta$, where (t, τ) are the gluing parameters of the domain. Still from the delicate pre-gluing procedure in [LT1] we can define the following stable orbits as “base point”

$$V_\alpha : \Sigma_\alpha \rightarrow M, \quad V_{(\alpha, t, \tau)} : \Sigma_{(\alpha, t, \tau)} \rightarrow M$$

where V_α is $W^{k,p}$ -close to V and $V_{(\alpha, t, \tau)}$ is the pre-gluing of V_α .

More precisely, we can define $V_\alpha = V \circ \psi_\alpha$, where $\psi_\alpha : \Sigma_\alpha \rightarrow \Sigma$ is diffeomorphism defined as follows. Fix a $r \ll \delta > 0$, define ψ_α to be identity on $\Sigma_\alpha \setminus \cup D_r(d_{l,k})$ and to be rotation of S^2 on each $D_\delta(d_{l,k})$ bringing $\alpha_{l,k}$ to $d_{l,k}$, where for simplicity D_r and D_δ denote half disc or disc according to whether $d_{l,k}$ is on the boundary or not. Since $r > 2\delta$, we can naturally assume the image of ψ_α restricting to D_δ is contained in D_r . So it is easy to smoothly extend ψ_α to all Σ_α . And when δ is small enough, ψ_α is smoothly close to identity. Thus, V_α is $W^{k,p}$ -close to V .

We can apply the very similar method in [LT1] to get the pre-gluing $V_{(\alpha, t, \tau)}$ of V_α . Recall that the domain $\Sigma_{(\alpha, t, \tau)}$ can be derived from Σ_α by gluing procedure listed in the last subsection. The difference is in that we consider gluing strips $L_{i,+\infty}$ and $L_{i+1,-\infty}$ of each pair of connecting main components instead of the annulus used by Liu-Tian. If there exist double point $d_{l,k}$ in the boundary, there also will be no trouble, we still can do the pre-gluing procedure firstly for a stable map defined on a larger domain, then we can restrict the resulting pre-gluing map to our original domain $\mathbb{R} \times [0, 1]$.

Also we can easily show that there exists a map

$$\psi_{(\alpha, t, \tau)} : (\Sigma_{(\alpha, t, \tau)}, z_1^*, \dots, z_k^*) \rightarrow (\Sigma, z_1, \dots, z_k),$$

which is injective outside all small discs contain double points. Then the full neighborhood of V is denoted by $\tilde{U}_\epsilon(V, \mathbf{H})$, which contains all points $(\Sigma_{(\alpha, t, \tau)}, \hat{V}, z'_1, \dots, z'_k)$ satisfying that \hat{V} is ϵ -close to $V \circ \psi_{(\alpha, t, \tau)}$ and each $\psi_{(\alpha, t, \tau)}(z'_i)$ is ϵ -close to z_i , where the parameters (α, t, τ) also vary in a δ -neighborhood.

Then as before we may extend the action of automorphism group Γ_V to the Γ_V -invariant set $\cup_{g \in \Gamma_V} g(\tilde{U}_\epsilon(V, \mathbf{H}))$, which is still denoted by $\tilde{U}_\epsilon(V, \mathbf{H})$, for simplicity, if no danger of confusion. Let $U_\epsilon(V, \mathbf{H}) = \tilde{U}_\epsilon(V, \mathbf{H})/\Gamma_V$, which can be regarded as a small neighborhood in \mathcal{W} . We just write $\mathcal{W} = \cup_{V \in \mathcal{PM}} U_{\epsilon_V}(V, \mathbf{H})$. Moreover, the $W^{k,p}$ -topology of \mathcal{W} can be generated by $U_\epsilon(V, \mathbf{H})$. This gives the \mathcal{W} an orbifold structure (c.f. Lemma 2.6 in [LT1]). Also Liu-Tian proved that the so-defined $W^{k,p}$ -topology is equivalent to the Floer-Gromov weak topology. This implies \mathcal{PM} is also Hausdorff and compact with respect to the strong $W^{k,p}$ topology.

Consequently, we can take a finite union of the covering of \mathcal{W} as

$$\{U_i = U_{\epsilon_{V_i}}(V_i, \mathbf{H}), i = 1, \dots, w\},$$

and we use \tilde{U}_i to denote its uniformizer with covering group Γ_i .

Then we can locally define orbifold bundles \mathcal{L}_i over \tilde{U}_i . For each $[V] \in U_i$, the fiber $\mathcal{L}_i|_{[V]}$ over $[V]$ consists of all elements of $L^{k-1,p}(\Lambda^{0,1}(V^*TM))$, $V \in [V]$ modulo equivalence relation induced by pull-back of sections coming from identification of the domains of V_i , where $\Lambda^{0,1}(V^*TM)$ is the bundle of $(0,1)$ -forms on Σ with respect to the complex structure on Σ and the given compatible almost complex structure J on (M, ω) . Then the local uniformizer $\tilde{\mathcal{L}}_i$ of \mathcal{L}_i is given by the union of $L^{k-1,p}(\Lambda^{0,1}(\tilde{V}_i^*TM))$, $\tilde{V}_i \in \tilde{U}_i$. The Γ_i also acts on $\tilde{\mathcal{L}}_i$ so that $\mathcal{L}_i = \tilde{\mathcal{L}}_i/\Gamma_i$. In this way, we can reinterpret the $\bar{\partial}_{J,H}$ -operator as a collection of Γ_i -equivariant sections of these local orbifold bundles (\mathcal{L}_i, U_i) .

More precisely, we will describe the construction in the rest of the section. For each $W \in \tilde{U}_\epsilon^D(V, \mathbf{H})$ or $\tilde{U}_\epsilon(V, \mathbf{H})$, the fiber is

$$\tilde{\mathcal{L}}^D(V)|_W = \tilde{\mathcal{L}}(V)|_W = \{\xi | \xi \in L^{k-1,p}(\Lambda^{0,1}(W^*TM))\},$$

where the $L^{k-1,p}$ -norm is measured with respect to the metric on the domain $\Sigma_W = \Sigma_{(\alpha, t, \tau)}$ induced by the gluing construction from the metric on Σ that is “spherical” on Σ_b and flat on Σ_m .

For fixed intersection pattern D , $\tilde{\mathcal{L}}^D(V)$ is a locally trivial Banach bundle over $\tilde{U}_\epsilon^D(V, \mathbf{H})$, and $\tilde{\mathcal{L}}(V)$ is locally trivial only when restricted to each stratum $\tilde{U}_\epsilon^D(V, \mathbf{H})$ of $\tilde{U}_\epsilon(V, \mathbf{H})$. So the topology of the bundle $\tilde{\mathcal{L}}(V)$, when restricted to each stratum of $\tilde{U}_\epsilon(V, \mathbf{H})$, is well-defined. We will not specify the topology right now, that will be done in the later gluing construction of virtual cycle. Instead, now we consider the more relevant following “sub-bundle” $\tilde{\mathcal{L}}_\delta^D(V)$ of $\tilde{\mathcal{L}}^D(V)$ without singularity, defined as

$$\tilde{\mathcal{L}}_\delta^D(V)|_W = \{\xi | \xi \in \tilde{\mathcal{L}}^D(V)|_W, \xi = 0 \text{ on each } D_\delta(d_{l,k}).\}$$

Let $\tilde{U}_{\epsilon, \delta}^D(V, \mathbf{H}) = \bar{\partial}_{J,H}^{-1}(\tilde{\mathcal{L}}_\delta^D(V))$, then we get a restricted bundle

$$\tilde{\mathcal{L}}_\delta^D(V) \rightarrow \tilde{U}_{\epsilon, \delta}^D(V, \mathbf{H}).$$

If $D \leq D_1$, then we can move the fiber of $\tilde{\mathcal{L}}_\delta^D(V)$ over some point in $\tilde{U}_{\epsilon, \delta}^D(V, \mathbf{H})$, by parallel transformation, into the fiber of $\tilde{\mathcal{L}}_{\delta_1}^{D_1}(V)$ over a neighborhood of the given point in $\tilde{U}_{\epsilon, \delta_1}^{D_1}(V, \mathbf{H})$, when $\delta_1 \ll \delta$. All these parallel transformations induce a topology for the union

$$\tilde{\mathcal{L}}^0(V) = \cup_{D, \delta} \tilde{\mathcal{L}}_\delta^D(V) \rightarrow \tilde{U}_\epsilon^0(V, \mathbf{H}) = \cup_{D, \delta} \tilde{U}_{\epsilon, \delta}^D(V, \mathbf{H}).$$

The Γ_V -actions on $\tilde{U}_\epsilon^D(V, \mathbf{H})$ and $\tilde{U}_\epsilon(V, \mathbf{H})$ can be lifted to the bundles via pull-back. Recall $U_\epsilon^D(V, \mathbf{H}) = \tilde{U}_\epsilon^D(V, \mathbf{H})/\Gamma_V$ and $U_\epsilon(V, \mathbf{H}) = \tilde{U}_\epsilon(V, \mathbf{H})/\Gamma_V$, we denote the orbifold bundles over them by $\mathcal{L}^D(V) = \tilde{\mathcal{L}}^D(V)/\Gamma_V$ and $\mathcal{L}(V) = \tilde{\mathcal{L}}(V)/\Gamma_V$, respectively.

With the above construction can see that the $\bar{\partial}_{J,H}$ -operator gives rise to a Γ_V -equivariant section of the bundle $\tilde{\mathcal{L}}(V) \rightarrow \tilde{U}_\epsilon(V, \mathbf{H})$, it is smooth on each stratum $\tilde{U}_\epsilon^D(V, \mathbf{H})$, and continuous when restricted to $\tilde{\mathcal{L}}^0(V) \rightarrow \tilde{U}_\epsilon^0(V, \mathbf{H})$. We still denote the section by $\bar{\partial}_{J,H}$. The zero sets $\bar{\partial}_{J,H}^{-1}(0)$ in $\tilde{U}_\epsilon^D(V, \mathbf{H})$ and $\tilde{U}_\epsilon(V, \mathbf{H})$ are just

$$\tilde{U}_\epsilon^D(V, \mathbf{H}) \cap P\mathcal{M}^D(J, H, \tilde{x}_-, \tilde{x}_+)$$

and

$$\tilde{U}_\epsilon(V, \mathbf{H}) \cap PM(J, H, \tilde{x}_-, \tilde{x}_+).$$

Thus, we consider \mathcal{W} as a *multi-fold* which is a partially smooth space—space with two topologies, which will be defined in the next section.

5 Abstract settings

All arguments in this section can be found in [LT1] and [M2], just for reader's convenience we give some related definitions and notations used later .

5.1 Partially smooth space and branched pseudomanifold

Definition 5.1 *A Hausdorff space Y is said to be partially smooth if it is the image of a continuous bijection $i_Y : Y_{sm} \rightarrow Y$, where Y_{sm} is a finite union of open disjoint subsets, each of which is a smooth Banach manifold.*

We consider the collection of all partially smooth spaces as objects of a category, and a morphism is a continuous map $f : Y \rightarrow X$ between two objects such that the induced map $f : Y_{sm} \rightarrow X_{sm}$ is smooth, say a partially smooth map. We see such Y is stratified, the strata that are open subsets of Y are called top strata.

Definition 5.2 *A pseudomanifold of dimension d is a compact partially smooth space Y such that a component Y_{sm} is an oriented smooth d -dimensional manifold which is mapped by i_Y onto a dense open subset Y^{top} of Y , and the dimensions of all other components of Y_{sm} are no larger than $d - 2$.*

We denote by $Y^{sing} = Y - Y^{top}$ for the image of those lower dimensional submanifolds. The following object is more general.

Definition 5.3 *A branched pseudomanifold Y of dimension d (without boundary) is a compact partially smooth space such that its components have at most dimension d . We denote the components of dimension d by M_i , those of dimension $d - 1$ by B_j , and write*

$$Y^{top} = \bigcup_i M_i, \quad B = \bigcup_j B_j, \quad Y^{sing} = Y - (Y^{top} \cup B).$$

Especially, we assume that for each i the set $M_i \cup_{j \in J_i} B_j$ ($j \in J_i$ if the closure of M_i in Y meets B_j) has the structure of smooth manifold with boundary which is compatible with its two topologies. And we assume $\bar{B}_j - B_j \subset Y^{sing}$. We call B the branched locus. A branched d -pseudomanifold with boundary is defined similarly, but some $d - 1$ dimensional components of ∂M_i are not any of the branched locus. We denote the union of such $d - 1$ dimensional components by ∂Y .

In order to construct the virtual cycle, we need a suitable labeling of the top components.

Definition 5.4 We say a branched pseudomanifold Y (with or without boundary) is labeled if its top components M_i are oriented and have positive rational labeling $\lambda_i \in \mathbb{Q}$ satisfying that for each $x \in B$, if we pick an orientation of $T_x B$, and divide the components M_i which have x in their closure into two groups I^+, I^- according to whether the chosen orientation on $T_x B$ agrees with the boundary orientation, then $\sum_{i \in I^+} \lambda_i = \sum_{j \in I^-} \lambda_j$.

In particular, if Y is of dimension zero, it is a collection of oriented labeled points, and there is no compatibility condition at the branch locus since it is empty.

If a compact branched pseudomanifold Y of dimension d is labeled as above, then McDuff [M2] in the following Lemma showed that it can be regarded as a (relative) cycle which represents a (relative) rational class.

Lemma 5.1 Let Y be a (closed, oriented) branched and labelled pseudomanifold of dimension d . Then every partially smooth map f from Y to a closed manifold X defines a rational class $f_*([Y]) \in H_d(X)$.

Proof. Let Z be a smooth manifold of dimension complementary to Y and $g : Z \rightarrow X$ be a smooth map. Then we can jiggle g so that it doesn't meet $f(Y^{sing})$ and so that it meets $\cup f(M_i)$ transversally in a finite number of points. We then define

$$f \cdot g = \sum_i \delta_i \lambda_i,$$

where $\delta_i = +1$ or -1 for $i \in I^+$ or I^- . One can check this number is independent of jiggling. Then we can define $f_*([Y])$ to be the unique rational class such that the intersection number

$$f_*([Y]) \cdot g_*([Z]) = f \cdot g, \quad \text{for all } g : Z \rightarrow X.$$

So we can look Y as a cycle. □

If Y is of dimension 1 with boundary, then from the way of labeling we see that the oriented number of its boundary is zero. More precisely, let $x \in \partial Y \cap M_i$, and denote by $\vec{v}_i \in T_x M$ the outward unit normal vector. This vector together with the orientation of M_i determines a sign

$$\delta_i = \begin{cases} +1, & \text{if } \vec{v}_i \text{ is positively oriented,} \\ -1, & \text{if } \vec{v}_i \text{ is negatively oriented.} \end{cases}$$

Then for each boundary point we define a number

$$\rho(x) = \sum_{x \in M_i} \delta_i \lambda_i.$$

D. Salamon proved the following result [S]

Lemma 5.2 Let (Y, λ) be a compact oriented, branched, and labeled 1-pseudomanifold with boundary. For each $x \in \partial M$ we have a rational number $\rho(x)$ defined above. Then

$$\sum_{x \in \partial M} \rho(x) = 0.$$

Another point is that, as much as possible, we are trying to avoid specifying exactly how the strata of Y_{sm} fit together. If Y is branched then one does need some information of the normal structure to the codimension 1 components, but one can often get away without any other restrictions. However, later we will need to consider the intersection of two partially smooth spaces, and in order for this to be well-behaved one does need more structure. This is the reason for the following definition.

Definition 5.5 *We will say that a partially smooth space Y has normal cones if every stratum S in Y has a neighborhood $N(S)$ in Y whose induced stratification is that of a cone bundle over a link. More precisely, there is a commutative diagram*

$$\begin{array}{ccc} N(S)_{sm} & \longrightarrow & N(S) \\ \pi' \downarrow & & \pi \downarrow \\ S_{sm} & \longrightarrow & S \end{array} \quad (6)$$

where the maps π' and π are oriented locally trivial fibrations with fiber equal to a cone over a link L . Here L is partially smooth, and the cone $C(L)$ is just the quotient $L \times [0, 1]/L \times 0$, stratified so that it is the union of the vertex (the image of $L \times 0$) with the product strata in $L_{sm} \times (0, 1]$. Moreover each stratum in $N(S)_{sm}$ projects onto the whole of S and so is a locally trivial bundle over S with fiber equal to a component of $C(S)_{sm}$. In particular, we identify S with the section of $N(S)$ given by the vertices of the cones.

This definition implies that any stratum S' whose closure intersects S must contain the whole of S in its closure. In fact, near S it must look like the cone over some stratum of L_{sm} . Any finite dimensional Whitney stratified space has this normal structure. For example, any finite dimensional orbifold has a stratification such that the isomorphism class of the automorphism group Γ_x at the point x is constant as x varies over each stratum, and it is easy to see that this stratification has normal cones as defined here. It is also not hard to use arguments similar to those in last subsection to show that the neighborhood \mathcal{W} of PM in the space of all stable maps has normal cones with respect to the fine stratification.

5.2 Multi-fold, multi-bundle, and multi-section

Now still following [M2], we introduce the concept of multi-fold which is a generalization of orbifold. It can be considered as atlas (or covering) of a space \mathcal{W} with two topologies that locally is an orbifold in the partially smooth category, in which the inclusions that relate one uniformizer to another in an orbifold are replaced by *fiber products* used by Liu-Tian, also they consider the fiber product as a certain kind of “global uniformizer”. We then show the definition of multi-sections of a multi-bundle over a multi-fold. All maps, spaces and group actions considered below are in this partially smooth category.

Suppose that a space \mathcal{W} is a finite union of open sets U_i , $i = 1, \dots, w$, each with uniformizers $(\tilde{U}_i, \Gamma_i, \pi_i)$ with the following properties. Each Γ_i is a finite group acting on \tilde{U}_i and the projection π_i is the composite of the quotient map $\tilde{U}_i \rightarrow \tilde{U}_i/\Gamma_i$

with an identification $\tilde{U}_i/\Gamma_i = U_i$. The inverse image in \tilde{U}_i of each stratum in U_i is an open subset of a (complex) Banach space on which Γ_i acts complex linearly. For simplicity, we will assume that Γ_i acts freely on the points of the top strata in \tilde{U}_i , and that the isomorphism class of the stabilizer subgroup $\text{Stab}_i(\tilde{x}_i)$ of Γ_i is fixed as \tilde{x}_i varies over a stratum of \tilde{U}_i , where the $\text{Stab}_i(\tilde{x}_i)$ is the subgroup of Γ_i that fixes \tilde{x}_i .

For each subset $I = \{i_1, \dots, i_p\}$ of $\{1, \dots, w\}$ set $U_I = \cap_{j \in I} U_j$, and $U_\emptyset = \emptyset$. Let $\mathcal{N} = \{I \mid U_I \neq \emptyset\}$. For $I \in \mathcal{N}$, denote $\Gamma_I = \prod_{j \in I} \Gamma_j$.

Definition 5.6 *The fiber product of those \tilde{U}_j , $j \in I$ is*

$$\tilde{U}_I = \{\tilde{x}_I = (\tilde{x}_j)_{j \in I} \mid \pi_j(\tilde{x}_j) = \pi_l(\tilde{x}_l) \in U_I, \text{ for all } j, l \in I\}$$

which is contained in $\prod_{j \in I} \tilde{U}_j$ with the two topologies induced from $\prod_{j \in I} \tilde{U}_j$.

Since it is easy to see Γ_I acts on \tilde{U}_I , the quotient $\tilde{U}_I/\Gamma_I \simeq U_I$. Denote the projection by $\pi_I : \tilde{U}_I \rightarrow U_I$. Note that \tilde{U}_I , U_I are also partially smooth spaces. And the isomorphism class of the stabilizer subgroup

$$\text{Stab}_I(\tilde{x}_I) = \prod_{j \in I} \text{Stab}_j(\tilde{x}_j)$$

of \tilde{x}_I in Γ_I is constant on each stratum, and trivial at points of top strata. Roughly speaking, we can consider the fiber product as a substitute for $\bigcap_{j \in I} \tilde{U}_j$. The topological structure can be understood in terms of notions *local component* and *desingularization* defined by Liu-Tian [LT2].

More precisely, Given a point $\tilde{x}_I \in \tilde{U}_I$, choose $i_0 \in I$ and a small open neighborhood \tilde{N} of $\tilde{x}_{i_0} \in \tilde{U}_{i_0}$. Then a neighborhood of \tilde{x}_I in \tilde{U}_I can be identified with the set

$$\{(\Upsilon_j \circ \iota_{ji_0}(\tilde{y}))_{i \in I} : \tilde{y} \in \tilde{N}, \Upsilon_{i_0} = id, \Upsilon_j \in \text{Stab}_j(\iota_{ji_0}(\tilde{y})), j \neq i_0\}.$$

It is a finite union of sets $\tilde{N}_\Upsilon(\tilde{x}_I)$, where each element Υ is in the group

$$\text{Stab}_{I-i_0}(\tilde{x}_I) = \prod_{j \in I-i_0} \text{Stab}_j(\tilde{x}_j).$$

In the partially smooth category, each such set is homeomorphic to \tilde{N} . It is clear that the germs at \tilde{x}_I of the sets $\tilde{N}_\Upsilon(\tilde{x}_I)$ are independent of the choices of i_0 and \tilde{N} , as is the isomorphism class of the “reduced” group $\text{Stab}'_I(\tilde{x}_I) = \text{Stab}_{I-i_0}(\tilde{x}_I)$. These germs are called the local components of \tilde{U}_I at \tilde{x}_I , denoted by $\langle \tilde{N}_\Upsilon(\tilde{x}_I) \rangle$. Since the points \tilde{x}_I in the top strata have trivial stabilizer groups, they only have a single local component. Then the “desingularization” \hat{U}_I of \tilde{U}_I is defined to be the union of all such local components

$$\hat{U}_I = \{(\tilde{x}_I, \langle \tilde{N}_\Upsilon(\tilde{x}_I) \rangle) : \tilde{x}_I \in \tilde{U}_I, \Upsilon \in \text{Stab}'_I(\tilde{x}_I)\}.$$

We can topologize the desingularization \hat{U}_I so that a germ of neighborhood contained in \hat{U}_I at the point $(\tilde{x}_I, \langle \tilde{N}_\Upsilon(\tilde{x}_I) \rangle)$ is homeomorphic to the local component $\langle \tilde{N}_\Upsilon(\tilde{x}_I) \rangle$ itself. So the projection

$$proj : \hat{U}_I \rightarrow \tilde{U}_I$$

is locally a homeomorphism onto its image. Since locally \widehat{U}_I is homeomorphic to the initial sets \tilde{U}_i , the extra singularities of \tilde{U}_I , introduced by constructing the fiber product, are of no trouble.

If $J = (j_1, \dots, j_q) \subset I = (i_1, \dots, i_p)$, there are two natural projections

$$\pi_J^I : \tilde{U}_I \rightarrow \tilde{U}_J = \tilde{U}_I / \Gamma_{I-J}, \quad \lambda_J^I : \Gamma_I \rightarrow \Gamma_J,$$

induced from the corresponding projection $\prod_{i_k \in I} \tilde{U}_{i_k} \rightarrow \prod_{j_l \in J} \tilde{U}_{j_l}$ such that $\pi_J \circ \pi_J^I = \iota_J^I \circ \pi_I$, where ι_J^I is the inclusion $U_I \hookrightarrow U_J$. We see that if $\pi_J^I(\tilde{y}_I) = \tilde{x}_J$ is in a top stratum, then $(\pi_J^I)^{-1}$ has $|\Gamma_{I-J}|$ points.

Let $\{\mathbf{V}_I\}$ be a open cover of \mathcal{W} such that $\mathbf{V}_I \subset U_I$ for each I . From the example 4.10 in [M2] we know that in general the sets \mathbf{V}_j , $j = 1, \dots, w$, no longer cover \mathcal{W} . Then we define $\tilde{\mathbf{V}}_I \subset \tilde{U}_I$ to be the inverse image of \mathbf{V}_I under the map $\pi : \tilde{U}_I \rightarrow U_I$. So Γ_I acts on $\tilde{\mathbf{V}}_I$ and we still write the quotient map as $\pi : \tilde{\mathbf{V}}_I \rightarrow \mathbf{V}_I \simeq \tilde{\mathbf{V}}_I / \Gamma_I$. We define the projection $\pi_J^I : \tilde{\mathbf{V}}_I \rightarrow \tilde{\mathbf{V}}_J$ as the restriction of the projection π_J^I above with domain $(\pi_J^I)^{-1}(\tilde{\mathbf{V}}_J)$.

Definition 5.7 *A multi-fold atlas for \mathcal{W} is a collection*

$$\tilde{\mathcal{V}} = \{(\tilde{\mathbf{V}}_I, \Gamma_I, \pi_J^I, \lambda_J^I), I \in \mathcal{N}\}.$$

The \mathcal{W} with such an atlas is called a multi-fold.

The motivation of considering such an atlas $\tilde{\mathcal{V}}$ or a subcover $\{\mathbf{V}_I\}$ of $\{U_I\}$ is in that in general (especially, in our application that \mathcal{W} is a neighborhood of stable moduli space PM) the suitable chosen $\{\mathbf{V}_i\}$ will have simpler overlaps rather than $\{U_i\}$ and when the sets U_i overlap too much there are no non-equivariant multi-sections which will be defined below. Since perturbation in the class of equivariant sections is not sufficient to realize regularity, we hope to obtain non-equivariant ones.

We now introduce the concept of *multi-bundle*. Let $\mathcal{E} = \bigcup_i E_i$ be a space with two topologies, and a map $p : \mathcal{E} \rightarrow \mathcal{W}$ with the property that each set $E_i = p^{-1}(U_i)$ has a local uniformizer $(\tilde{E}_i, \Gamma_i, \Pi_i)$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{E}_i & \xrightarrow{\Pi_i} & E_i \\ \tilde{p} \downarrow & & p \downarrow \\ \tilde{U}_i & \xrightarrow{\pi_i} & U_i. \end{array} \quad (7)$$

And the map $\tilde{p} : \tilde{E}_i \rightarrow \tilde{U}_i$ is required to be Γ_i -equivariant such that its restriction to each stratum of \tilde{U}_i is a locally trivial vector bundle. Then the fiber $\tilde{F}(\tilde{x}_i)$ of \tilde{p} at each point \tilde{x}_i is a vector space, but with no natural identification of two different fibers if they are over points in different strata. We denote by \tilde{E}_I the restriction to $\tilde{\mathbf{V}}_I$ of the fiber product of the \tilde{E}_i , $i \in I$, over \tilde{U}_I .

We suppose that the orbifold structure on \mathcal{W} can lift to one on \mathcal{E} , then \mathcal{E} had the same local structure as \mathcal{W} . As above, we define a multi-fold atlas $\tilde{\mathcal{E}} = \{\tilde{E}_I\}$ for \mathcal{E} . More precisely, the elements of the fiber $\tilde{F}(\tilde{x}_I)$ of \tilde{E}_I at $\tilde{x}_I \in \tilde{V}_I$ are $(\tilde{x}_i, \tilde{v}_i)_{i \in I}$, where $\tilde{v}_i \in \tilde{F}(\tilde{x}_i)$, and for all $j, l \in I$,

$$\Pi_j(\tilde{v}_j) = \Pi_l(\tilde{v}_l) \in E_I.$$

Also we can get the desingularization

$$\widehat{E}_I \rightarrow \widehat{\mathbf{V}}_I$$

of $\tilde{E}_I \rightarrow \tilde{\mathbf{V}}_I$, which is an honest vector bundle since locally it has the same structure as the maps $\tilde{E}_i \rightarrow \tilde{U}_i$, $i \in I$. Thus $\tilde{E}_I \rightarrow \tilde{\mathbf{V}}_I$ is a finite union of vector bundles with each fiber being a finite union of vector spaces.

Definition 5.8 *We say that the map $\tilde{p}: \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{V}}$ constructed above is a multi-bundle with respect to the multi-fold atlas defined in the Definition 5.7.*

Now, we come to the definition of **multi-section** of the multi-bundle $\tilde{p}: \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{V}}$. Intuitively we consider it as a *compatible collection* $\{\tilde{s}_I\}$ of multi-valued sections. That is to say, we have a nonempty finite subset $\tilde{s}_I(\tilde{x}_I)$ of $\tilde{F}_I(\tilde{x}_I)$ for each point $\tilde{x}_I \in \tilde{\mathbf{V}}_I$. In our application, it is enough to consider multi-sections that are single-valued when lifted to the desingularization $\widehat{E}_I \rightarrow \widehat{\mathbf{V}}_I$. We assume that there exist sections $\hat{s}_I: \widehat{\mathbf{V}}_I \rightarrow \widehat{E}_I$ such that

$$\tilde{s}_I(\tilde{x}_I) = \{proj \circ \hat{s}_I(\hat{x}_I, \langle \tilde{N}_\Upsilon(\hat{x}_I) \rangle) : \Upsilon \in \text{Stab}'_I(\hat{x}_I)\}.$$

Note that when $I = \{j\}$ the section $\tilde{s}_j = \hat{s}_j$ is a single-valued section of the bundle $\tilde{E}_j \rightarrow \tilde{U}_j$ which is maybe non-equivariant. And when \tilde{x}_I is in the top stratum the set $\tilde{s}_I(\tilde{x}_I)$ has only one element in $\tilde{F}_I(\tilde{x}_I)$.

Let $J \subset I$, recall there is projection $\Pi_J^I: \tilde{E}_I \rightarrow \tilde{E}_J$. For a multi-section \tilde{s}_J we define its pullback $(\Pi_J^I)^{-1}(\tilde{s}_J)$ to be a multi-valued section so that each \tilde{x}_I is associated with the full inverse image $(\Pi_J^I)^{-1}(\tilde{s}_J(\tilde{x}_J))$ where $\tilde{x}_J = \pi_J^I(\tilde{x}_I)$. Then the *compatibility* condition requires that

$$\tilde{s}_I|_{(\pi_J^I)^{-1}(\tilde{\mathbf{V}}_J)} = (\Pi_J^I)^* \tilde{s}_J. \quad (8)$$

It is easy to see that if for all local bundles $\tilde{p}: \tilde{E}_i \rightarrow \tilde{U}_i$ a family of Γ_i -equivariant sections \tilde{s}_i are compatible, then the multi-bundle $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{V}}$ has a multi-section. In our application the $\bar{\partial}_{J,H}$ operator is such an example. However, because of the appearance of multi-covered spheres with negative Chern class, perturbations of the Cauchy-Riemann equations in the equivariant class can not regularize the (stable) moduli space. Therefore, we have to consider a method of extending non-equivariant multi-valued sections of bundles $\tilde{E}_i \rightarrow \tilde{U}_i$ to multi-sections of $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{V}}$, such that they can be regarded as global perturbations of the $\bar{\partial}_{J,H}$ operator. Actually, if we choose a “good” subcover $\{\mathbf{V}_I\}$ of $\{U_I\}$ which do not overlap too much, the extension is possible. The following lemma shows the existence of such a good $\{\mathbf{V}_I\}$, it is taken from [LT1] and [M2]. The notation $A \subset\subset B$ means that the closure of A is contained in B .

Lemma 5.3 *Given a finite open covering $\{U_i, i = 1, \dots, N\}$ of a compact subset PM of \mathcal{W} and U_I is defined as above, then there are open subsets $U_i^0 \subset\subset U_i$ and $\mathbf{V}_I \subset U_I$ satisfying (i) $PM \subset \cup_i U_i^0$, $PM \subset \cup_I \mathbf{V}_I$; (ii) If $i \notin I$, $U_i^0 \cap \mathbf{V}_I = \emptyset$; (iii) if $\mathbf{V}_I \cap \mathbf{V}_J \neq \emptyset$ then $I \subset J$ or $J \subset I$.*

Proof. For $n = 0, 1, \dots, N$ choose open coverings $\{U_i^n\}, \{W_i^n\}$ of PM satisfying

$$U_i^0 \subset\subset W_i^1 \subset\subset U_i^1 \subset\subset W_i^2 \subset\subset \dots \subset\subset U_i^N = U_i.$$

Then, let $\kappa = |I|$, we define

$$\mathbf{V}_I = W_I^\kappa - \bigcup_{J: |J| > \kappa} \text{Closure of } U_J^{\kappa+1}.$$

All properties hold obviously. \square

By shrinking \mathcal{W} , we may suppose $\mathcal{W} = \cup_i U_i^0 = \cup_I \mathbf{V}_I$. For a multi-bundle $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{V}}$ as above, we assume that for some j we have a section $\sigma(j) : \tilde{U}_j \rightarrow \tilde{E}_j$ of the bundle $\tilde{E}_j \rightarrow \tilde{U}_j$ with support in \tilde{U}_j^0 . Then, for each I if $j \notin I$, we can define $\tilde{s}(j)_I$ to be the zero section of $\tilde{E}_I \rightarrow \tilde{\mathbf{V}}_I$ and, otherwise, we can define $\tilde{s}(j)_I$ to be the restriction to $\tilde{\mathbf{V}}_I$ of the pullback to \tilde{U}_I of the graph of $\sigma(j)$. This induces a multi-section $\tilde{s}(j)$ of $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{V}}$, $j = 1, 2, \dots, w$. That is to say, choosing suitable covers $\{U_i^0\}$ and $\{\mathbf{V}_I\}$ as in the lemma 5.3, it is always possible to construct a multi-section. In applications, we need the section $\sigma(j)$ satisfies some generic conditions and require the boundary of the support of $\sigma(j)$ is a union of strata (c.f. [M2]). In the next section we will use a form of $\tilde{s} = \bar{\partial}_{J,H} + \sum_j \tilde{s}(j)$ with generic perturbation term to define a Fredholm multi-section which is transversal to zero section and derive the virtual moduli cycle.

5.3 Constructing the virtual cycle

Here, we will construct a branched pseudomanifold from the multi-section defined above, that is just the virtual cycle.

In the following, we always take the cover $\{\mathbf{V}_I\}$ as in the lemma 5.3. In the partially smooth category we also have a similar definition of *Fredholm system* as the Definition 1.1. For each I , we denote by $gr(\tilde{s}_I)$ for the graph of the section which is the union $\bigcup_{\tilde{x}_I \in \tilde{\mathbf{V}}_I} \tilde{s}_I(\tilde{x}_I)$, we always require the graph is an object and the projection $gr(\tilde{s}_I) \rightarrow \tilde{\mathbf{V}}_I$ is a morphism in the partially smooth category. This amounts to requiring that for each $j \in I$ the $gr(\tilde{s}_j)$ has a stratification which is compatible with the projection to $\tilde{\mathbf{V}}_j$ and is preserved by the action of the group Γ_j such that the rank of the stabilizer is constant on strata (after refining of the stratification of $\tilde{\mathbf{V}}_I$). Let $\tilde{\mathcal{E}}, \tilde{\mathcal{V}}, \mathcal{W}$ are defined as above, \tilde{s} is a multi-section of $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{V}}$

Definition 5.9 A system $(\tilde{\mathcal{E}}, \tilde{\mathcal{V}}, \mathcal{W}, \tilde{s}, \mathcal{N})$ is called to be a *transverse Fredholm system* with index d if the following hold

- 1) For each $I \in \mathcal{N}$, \tilde{s}_I is a Fredholm section, $gr(\tilde{s}_I)$ intersects transversally with the zero section of $\tilde{E}_I \rightarrow \tilde{\mathbf{V}}_I$ in a d -dimensional pseudomanifold \tilde{Z}_I , which is also stratified with top stratum coincides with its intersections with the top stratum of $\tilde{\mathbf{V}}_I$.
- 2) The union $Z_{\mathcal{W}} = \bigcup_I \pi_I(\tilde{Z}_I)$ is compact in \mathcal{W} . \tilde{s} is called a Fredholm multi-section.
- 3) If the top strata of \tilde{Z}_I are oriented, then all orientations are preserved under the partially defined projections π_J^I .

When we generically perturb the section over the stratified spaces we work inductively over the strata. If \tilde{s}_I is perturbed over one stratum so that it is transverse to the zero section there, we need to extend this perturbation to nearby strata, and hence we need information on how that strata fit together. While McDuff [M2] showed that if there are suitable normal cones defined in Definition 5.5, for obtaining a Fredholm multi-section, it is sufficient to perturb $gr(\tilde{s})_I$ so that the intersection of each stratum S' of $gr(\tilde{s})_I$ with a stratum S of $\tilde{\mathbf{V}}_I$ is transverse inside $\tilde{p}^{-1}(S)$ which is a stratum of \tilde{E}_I . In application, Liu-Tian [LT1] also proved that they can get a transverse Fredholm multi-section $\tilde{\partial}_{J,H}^\nu$ of some multi-bundle over the space of stable (J, H) maps by choosing perturbations in finite vector spaces R_I .

Then we should assemble all the local zero sets \tilde{Z}_I into a closed branched labeled pseudomanifold Y which projects onto $Z_{\mathcal{W}}$. For each I , we choose a manifold with boundary $\mathbf{V}'_I \subset \mathbf{V}_I$ so that all \mathbf{V}'_I cover \mathcal{W} . Let $\tilde{Y}_I = \tilde{Z}_I \cap \pi_I^{-1}(\mathbf{V}'_I)$. Here \tilde{Y}_I is a closed pseudomanifold and is given the obvious first topology, and the components of $(\tilde{Y}_I)_{sm}$ either lie on its boundary (*i.e.* in $\pi_I^{-1}(\partial\mathbf{V}'_I)$), or are the intersections of strata in \tilde{Z}_I with its interior. In order for there to be such a stratification, it suffices that the stratification of \mathcal{W} can be refined so that each boundary $\partial(\mathbf{V}'_I)$ is a union of strata. Thus the boundaries $\partial(\mathbf{V}'_I)$ must intersect transversally and be in general position with respect to the original strata of \mathcal{W} . Strictly speaking, one can arrange this for general \mathcal{W} only in the presence of suitable normal cones. If one can find such sets \tilde{Y}_I we will say that they form a shrinking of the \tilde{Z}_I . Thus, we have stratified the \tilde{Y}_I so that their only codimension 1 strata lie in the boundary $\partial(\mathbf{V}'_I)$. These will correspond to the branching locus of Y .

We then construct a topological space Y such that there are continuous maps

$$\coprod_I \tilde{Y}_I \longrightarrow Y \longrightarrow Z_{\mathcal{W}},$$

where Y consists of points $[y]$ which are the equivalence classes under the equivalence relation given by

$$\tilde{x}_I \sim \tilde{y}_J, \quad \text{if } J \subset I, \quad \pi_J^I(\tilde{x}_I) = \tilde{y}_J.$$

The first topology on Y is the quotient topology, and the strata of its second (smooth) topology are formed by the images of strata in the \tilde{Y}_I , subdivided if necessary. It will be convenient to refine the stratifications of the \tilde{Y}_I so that the projections $q_I : \tilde{Y}_I \rightarrow Y$ take strata to strata. This introduces new codimension 1 strata in the \tilde{Y}_I coming from the boundaries of the \tilde{Y}_K for $I \subset K$.

In order to define a cycle, we still need a suitable labeling of the top components, *i.e.* to define a positive rational labeling function. For a top stratum S lying the image of $q_I : \tilde{Y}_I \rightarrow Y$, we define

$$\lambda_I([y]) = \frac{|\{q_I^{-1}([y])\}|}{|\Gamma_I|}, \quad [y] \in S,$$

where $|\cdot|$ denotes the cardinality of a set or group. McDuff proved the following result, for the reader's convenience we restate her proof.

Proposition 5.1 *Under the setting as above the labeling function λ_I descends to a function λ on Y , and (Y, λ) can be regarded as a compact labeled branched pseudo-manifold as is defined in Definition 5.4.*

Proof. Since for $J \subset I$ and $y \in q_I(\tilde{Y}_I) \cap q_J(\tilde{Y}_J)$, \tilde{s}_I is the pull-back of \tilde{s}_J over $(\pi_J^I)^{-1}(\tilde{\mathbf{V}}_J)$ such that

$$|\{q_I^{-1}([y])\}| = |\{q_J^{-1}([y])\}| \cdot |\Gamma_{I-J}|,$$

and $|\Gamma_I| = |\Gamma_J| \cdot |\Gamma_{I-J}|$, we see that $\lambda_I([y]) = \lambda_J([y])$, thus we get a function λ on Y .

To check the branching condition in Definition 5.4 on the codimension 1 components of Y , we have to understand how the generating equivalence $\tilde{x}_I \sim \tilde{y}_J$, $J \subset I$, effect on such condition. Note first that a class $[y]$ lies in a codimension 1 stratum B_l in Y , if for one K the representatives of $[y]$ in \tilde{Y}_K lie on the boundary stratum B_K and if for all other J the representatives in \tilde{Y}_J lie in the interior of \tilde{Y}_J . We call the side of B_K that meets the interior of \tilde{Y}_K the positive side, and denote by $\{S_{K,\alpha}^+\}$ for the set of all top strata in \tilde{Y}_K whose closures contain representative of $[y]$. For all other J , we denote by $\{S_{J,\alpha}^\pm\}$ for the set of all top strata in \tilde{Y}_J containing representative of $[y]$, with sign assigned in the obvious way. By the Lemma 5.3, the intersection $I_y = J \cap K$ is nonempty and everything is essentially pulled back from \tilde{V}_{I_y} . For instance, two strata in $\{S_{J,\alpha}^+\}$ are identified in Y if and only if they have the same image in \tilde{V}_{I_y} . Therefore, we only consider the effect of inclusions of the form $I_y \subset I$ on the branching condition. We distinguish three cases: i) $K \neq I_y$ or I ; ii) $K = I$; iii) $K = I_y$.

For case i), assume J is either I_y or I , there is a bijective correspondence between the components $\{S_{J,\alpha}^+\}$ and $\{S_{J,\alpha}^-\}$ on both sides that commutes with the identifications coming from the equivalence \sim . So the labelings are the same on both sides.

For case ii), each stratum in $\{S_{K,\alpha}^+\}$ is mapped bijectively by $\pi_{i_Y}^k$ onto a stratum in $\{S_{I_y,\alpha}^+\}$, and each of the latter strata is covered exactly $|\Gamma_{K-I_y}|$ times. Also both sides of B in Y are the same and no real branching occurs either.

For case iii), the situation is different. The components in $\{S_{I,\alpha}^+\}$ are identified via π_K^I with components in $\{S_{K,\alpha}^+\}$, while no identification is put on the components $\{S_{I,\alpha}^+\}$ on the other side. Since π_K^I is $|\Gamma_{I-K}|$ to 1, $|\Gamma_{I-K}|$ components on the negative side of B in Y will correspond to each component on the positive side. One can check that the sum of labels on both sides is the same. \square

Recall that such a d -dimensional branched pseudomanifold Y can be considered as a cycle in the sense that every partially smooth map f from Y to a closed manifold X defines a rational class $f_*([Y]) \in H_d(X)$, say *virtual cycle* (c.f. Lemma 5.1 or [M2]).

6 Transversality.

Let us come back to our problem. In order to apply arguments showed above to our case of moduli space of stable connecting orbits, we need firstly construct a transverse Fredholm system from $\mathcal{PM}(\tilde{x}_-, \tilde{x}_+) \subset \mathcal{B}(\tilde{x}_-, \tilde{x}_+)$ as defined in the Definition 5.9. Because of the possibility of bubbling-off multiple covered holomorphic spheres with negative first Chern class in a connecting orbit $u \in \mathcal{M}(\tilde{x}_-, \tilde{x}_+)$, even for generic

pair (J, H) , the linearized map D_u in general does not be surjective. However, one can show that under a locally non-equivariant perturbation of the $\bar{\partial}_{J,H}$ -operator constricted in a small neighborhood U_ϵ in the ambient space $\mathcal{B}(\tilde{x}_-, \tilde{x}_+)$, denoted by $\bar{\partial}_{J,H} + \nu$, the regularity will hold. For instance, for defining the classical Floer homology for some chain complex generated by nondegenerate periodic solutions of Hamiltonian flow, Liu-Tian (c.f. section 3 in [LT1]) showed the full details to realize the regularization by non-equivariant perturbations, and define the local virtual moduli space \mathcal{M}'_ϵ which has the expected dimension. Since the only difference between in our case and in the classical one is that the boundary condition (2) is related to a symplectomorphism ϕ , which may not be identity, *i.e.* the solutions x_\pm are unnecessarily periodic, and this difference has little effect on all the process of construction, we then only give a sketchy and suggestive description below.

For any two $\tilde{x}_-, \tilde{x}_+ \in \text{Crit}(F)$, we suppose that the unparameterized stable orbit $[V]$ connecting \tilde{x}_- and \tilde{x}_+ . Recall $(\tilde{U}_\epsilon^D(V, \mathbf{H}), \Gamma_V, \pi_V)$ is a local uniformizer for $[V]$ in \mathcal{W}^D , and denote $\tilde{U}_\epsilon(V, \mathbf{H}) = \cup_D \tilde{U}_\epsilon^D(V, \mathbf{H})$ with automorphism group Γ_V , and we saw that $U_\epsilon(V, \mathbf{H}) = \tilde{U}_\epsilon(V, \mathbf{H})/\Gamma_V$ can be regarded as a small neighborhood of $[V]$ in $\mathcal{W} = \bigcup_{[V] \in P\mathcal{M}} U_{\epsilon_V}(V, \mathbf{H})$ which is a covering of $P\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$.

We then define locally the bundles $\tilde{E}^D(V)$ and $\tilde{E}(V)$ over $\tilde{U}_\epsilon^D(V, \mathbf{H})$ and $\tilde{U}_\epsilon(V, \mathbf{H})$ as follows. For each $\hat{V} \in \tilde{U}_\epsilon^D(V, \mathbf{H})$ or $\tilde{U}_\epsilon(V, \mathbf{H})$, the fiber is

$$\mathcal{F}_{\hat{V}} = \{\xi | \xi \in L^{k-1,p}(\Lambda^{0,1}(\hat{V}^*TM))\},$$

where the $L^{k-1,p}$ norm is suitably introduced by gluing process from the metric on the domain. We can see that for fixed D , $\tilde{E}^D(V)$ is a locally trivial Banach bundle over $\tilde{U}_\epsilon^D(V, \mathbf{H})$. So the topology of $\tilde{E}(V)$ is well-defined when restricted to each stratum of $\tilde{U}_\epsilon^D(V, \mathbf{H})$. As we claimed before, the $\bar{\partial}_{J,H}$ -operator can be considered as a Γ_V -equivariant section of the bundle $\tilde{E}(V) \rightarrow \tilde{U}_\epsilon(V, \mathbf{H})$, it is smooth on each stratum $\tilde{U}_\epsilon^D(V, \mathbf{H})$. The zero sets $\bar{\partial}_{J,H}^{-1}(0)$ in $\tilde{U}_\epsilon^D(V, \mathbf{H})$ and $\tilde{U}_\epsilon(V, \mathbf{H})$ are projected to

$$P\mathcal{M}^D(\tilde{x}_-, \tilde{x}_+) \cap U_\epsilon^D(V, \mathbf{H}) \subset \mathcal{B}^D(\tilde{x}_-, \tilde{x}_+)$$

and

$$P\mathcal{M}(\tilde{x}_-, \tilde{x}_+) \cap U_\epsilon(V, \mathbf{H}) \subset \mathcal{B}(\tilde{x}_-, \tilde{x}_+),$$

respectively.

We first consider those $[V]$ with fixed intersection pattern D . We can define a coordinate chart of $\tilde{U}_\epsilon^D(V, \mathbf{H})$ and a trivialization of $\tilde{E}^D(V)$. First, for fixed parameter α with intersection pattern D , we assume $x_{l,j}$ is the added marked point of a component Σ_α^l of Σ_α , and we denote the tangent space of $\mathbf{H}_{l,j}$ at $x_{l,j}$ by $H_{l,j}$. Recall the fact that we are restricting to a slice for the action of reparametrization group, we define the space

$$L^{k,p}(V_\alpha^*TM, H) = \{\xi = \xi_\alpha | \xi \in L^{k,p}(V_\alpha^*TM), \xi_l(x_{l,j}) \in H_{l,j}\},$$

and the set

$$W_\epsilon^\alpha = \{\xi \in L^{k,p}(V_\alpha^*TM, H), \|\xi\|_{k,p} < \epsilon\}.$$

It is a smooth coordinate chart for $\tilde{U}_{\epsilon_1}^\alpha(V_\alpha, \mathbf{H})$ near V_α for $\epsilon \ll \epsilon_1$. So $W_\epsilon = \cup_{\alpha \in D} W_\epsilon^\alpha$ is the coordinate chart for $\tilde{U}_{\epsilon_1}^D(V, \mathbf{H})$. In fact, if we denote

$$\Lambda_\epsilon = \{\alpha \mid \alpha \in D, \|\alpha\| < \epsilon\},$$

the W_ϵ has a splitting $W_\epsilon^{\alpha=0} \times \Lambda_\epsilon$ which can be regarded as the local coordinate of $\tilde{U}_{\epsilon_1}^D(V, \mathbf{H})$.

As the standard method (c.f. [MS][LT1]), we can get a trivialization of the bundle $\tilde{E}^\alpha(V)$ and $\tilde{E}^D(V)$, we denote the trivialization by

$$\gamma_D : \tilde{U}_\epsilon^D(V, \mathbf{H}) \times L^{k-1,p}(\Lambda^{0,1}(V^*TM)) \rightarrow \tilde{E}^D(V).$$

Under these local charts and trivialization, for fixed α , we can write the section

$$\bar{\partial}_{J,H} : \tilde{U}_\epsilon^\alpha(V, \mathbf{H}) \rightarrow \tilde{E}^\alpha(V)$$

as a nonlinear map

$$S_\alpha : W_\epsilon^0 \rightarrow L^{k-1,p}(\Lambda^{0,1}(V^*TM)).$$

Then the problem of transversality is to consider whether its linearized map is surjective. In particular, if ϵ is sufficiently small, we only need to componentwise deal with the operator

$$T_l = DS_{\alpha=0}(0) : L^{k,p}(V_l^*TM, H) \rightarrow L^{k-1,p}(\Lambda^{0,1}(V_l^*TM)),$$

which is a linear elliptic operator, it is also a Fredholm operator which can be proved via standard arguments (c.f.[MS]). While it is not surjective in general due to the appearing of multiple covered bubble sphere with negative first Chern class. However, the cokernel $K_l = K_l(V)$ of T_l is finite dimensional.³ Let $K = \oplus_{l=1}^L K_l$, then $L^{k-1,p}(\Lambda^{0,1}(V^*TM)) = K \oplus \text{Im}(T)$. On each main component, for generic pair (J, H) , T_m is surjective, so we only deal with bubble components. Then Liu-Tian in [LT1] showed the method of enlarging the domain of T to realize surjectivity. Roughly speaking, we can define a finite dimensional vector space $R \subset L^{k-1,p}(\Lambda^{0,1}(V^*TM))$ such that the new linear map

$$T \oplus \mathbf{I} : L^{k,p}(V_l^*TM, H) \oplus R \rightarrow L^{k-1,p}(\Lambda^{0,1}(V_l^*TM))$$

is surjective and the kernel of it is the same as the kernel of T , where $\mathbf{I} : R \rightarrow L^{k-1,p}(\Lambda^{0,1}(V_l^*TM))$ is inclusion.

Then we can extend R over $\tilde{U}_\epsilon^0(V, \mathbf{H})$ and $\tilde{U}_\epsilon^D(V, \mathbf{H})$ with $D = D(f)$. Thus when $\|\alpha\|$ are small enough, we can get $R(V) \subset L^{k-1,p}(\Lambda^{0,1}(V_\alpha^*TM))$. If $\dim R = r$, then we obtained a r -dimensional vector bundle R over $\tilde{U}_\epsilon^D(V, \mathbf{H})$. Also we have a surjective linear map

$$T_\alpha \oplus \mathbf{I}_\alpha : L^{k,p}(V_\alpha^*TM, H) \oplus R(V_\alpha) \rightarrow L^{k-1,p}(\Lambda^{0,1}(V_\alpha^*TM)).$$

³For later extending K_l to a vector bundle over $\tilde{U}_\epsilon(V, \mathbf{H})$, we may assume its vectors vanish at each bubble point (c.f.[LT1]).

Consequently, by implicit function theorem we know that the following moduli space

$$\widetilde{\mathcal{M}}_{R,\epsilon}^D(\tilde{x}_-, \tilde{x}_+) = \{\hat{V} \mid \hat{V} \in \widetilde{U}_\epsilon^D(V, \mathbf{H}), \bar{\partial}_{J,H}\hat{V} \in R\}$$

is a smooth manifold of dimension $r + \mu_{rel}(\tilde{x}_-, \tilde{x}_+) - 1$.

When we consider the changes of the topological type of the domains, we use the gluing parameter (t, τ) introduced in subsection 4.2. Similarly, for a fixed parameter (α, t, τ) , we can define $W_\epsilon^{(\alpha, t, \tau)}$ as a coordinate chart of $\widetilde{U}_\epsilon^{(\alpha, t, \tau)}(V, \mathbf{H})$, the trivialization of the bundle $\tilde{E}^{(\alpha, t, \tau)} \rightarrow \widetilde{U}_\epsilon^{(\alpha, t, \tau)}(V, \mathbf{H})$, and under the coordinate and trivialization we can regard the $\bar{\partial}_{J,H}$ section of this bundle as a nonlinear map

$$S_{(\alpha, t, \tau)} : W_\epsilon^{(\alpha, t, \tau)} \rightarrow L^{k-1, p}(\Lambda^{0,1}(V_{(\alpha, t, \tau)}^* TM)),$$

whose linearized map

$$T_{(\alpha, t, \tau)} = DS_{(\alpha, t, \tau)}(0) : L^{k, p}(V_{(\alpha, t, \tau)}^* TM, H) \rightarrow L^{k-1, p}(\Lambda^{0,1}(V_{(\alpha, t, \tau)}^* TM))$$

is a Fredholm operator. And we can define a finite dimensional subspace $R_{(\alpha, t, \tau)} = R(V_{(\alpha, t, \tau)}) \subset L^{k-1, p}(\Lambda^{0,1}(V_{(\alpha, t, \tau)}^* TM))$ and embedding \mathbf{I} such that the operator

$$T \oplus \mathbf{I} : L^{k, p}(V_{(\alpha, t, \tau)}^* TM, H) \oplus R_{(\alpha, t, \tau)} \rightarrow L^{k-1, p}(\Lambda^{0,1}(V_{(\alpha, t, \tau)}^* TM))$$

Then the critical thing is to show that when the parameters change in a small neighborhood

$$\Lambda_\delta = \{(\alpha, t, \tau) \mid \|(\alpha, t, \tau)\| < \delta\},$$

one can still find $R_{(\alpha, t, \tau)}$ and a partially smooth family of embedding $\mathbf{I}_{(\alpha, t, \tau)}$ such that the linearized operator

$$T_{(\alpha, t, \tau)} \oplus \mathbf{I}_{(\alpha, t, \tau)} : L^{k, p}(V_{(\alpha, t, \tau)}^* TM, H) \oplus R_{(\alpha, t, \tau)} \rightarrow L^{k-1, p}(\Lambda^{0,1}(V_{(\alpha, t, \tau)}^* TM))$$

is surjective, *i.e.* with uniform estimates for its right inverse as (α, t, τ) varies in the sufficiently neighborhood Λ_δ .

In order to get the desired uniform estimate, Liu-Tian (c.f. section 3 in [LT1]) used some exponential weighted equivalent norms $\|\cdot\|_{\chi; k, p}$ on $L^{k, p}(V_{(\alpha, t, \tau)}^* TM, H)$ and $L^{k-1, p}(\Lambda^{0,1}(V_{(\alpha, t, \tau)}^* TM))$. The same argument can apply to our case, and we can similarly prove that there exists a right inverse G of $T_{(\alpha, t, \tau)} \oplus \mathbf{I}_{(\alpha, t, \tau)}$ and a constant $c = c(V)$ depending only on V such that for sufficiently small δ , $(\alpha, t, \tau) \in \Lambda_\delta$, we have

$$\|G_{(\alpha, t, \tau)}\xi\|_{\chi; k, p} < c(V)\|\xi\|_{\chi; k, p}$$

for $\forall \xi \in L^{k-1, p}(\Lambda^{0,1}(V_{(\alpha, t, \tau)}^* TM))$. That is equivalent to say, if we shrink $\widetilde{U}_\epsilon(V, \mathbf{H})$ to be a sufficiently small neighborhood, then there exists a uniformly bounded family of right inverses to $T_{\hat{V}} \oplus \mathbf{I}_{\hat{V}}$ as \hat{V} varies in $\widetilde{U}_\epsilon(V, \mathbf{H})$. And in this small neighborhood $\widetilde{U}_\epsilon(V, \mathbf{H})$, we can identify $R_{(\alpha, t, \tau)}$ with $R_V = R_{(0,0,0)}$.

Then we can define

$$\mathcal{O} = \mathcal{O}_{\widetilde{U}_\epsilon(V, \mathbf{H})} = \widetilde{U}_\epsilon(V, \mathbf{H}) \times R_V$$

and a projection

$$P : \mathcal{O} \rightarrow \tilde{U}_\epsilon(V, \mathbf{H}),$$

for the pullback bundle $P^*(\tilde{E}_V) \rightarrow \tilde{U}_\epsilon(V, \mathbf{H}) \times R_V$, we can construct a section as⁴

$$s(\hat{V}, \nu) = \bar{\partial}_{J,H}(\hat{V}) + \mathbf{I}_{\hat{V}}(\nu), \quad (9)$$

where $\nu \in R_V$. From the construction above, its linearized operator is surjective at all points $(\hat{V}, \nu) \in \mathcal{O}$. Then by using some variant of gluing as in subsection 3.3 of [LT1] one can see that the zero set $s^{-1}(0)$ is a open partially smooth pseudomanifold of dimension $r + \mu_{rel}(\tilde{x}_-, \tilde{x}_+)$, whose components are the intersections of $s^{-1}(0)$ with each stratum $\tilde{U}_\epsilon^D(V, \mathbf{H})$. So we obtain a new transverse Fredholm system $(P^*(\tilde{E}_V), \mathcal{O}, s)$. Since R_V is finite dimensional, the Sard-Smale theorem (c.f. [MS]) says that for generic $\nu \in R_V$ we have a transverse Fredholm section

$$s^\nu : \tilde{E}_V \rightarrow \tilde{U}_\epsilon(V, \mathbf{H}),$$

satisfying $s = P^*(s^\nu)$. Then we denote its zero set by $\tilde{Z}_V^\nu = (s^\nu)^{-1}(0)$ and so we have

Proposition 6.1 *For generic $\nu \in R_V$, the section s^ν is transverse Fredholm and its zero set \tilde{Z}_V^ν has the structure of an open pseudomanifold of dimension $d = \mu_{rel}(\tilde{x}_-, \tilde{x}_+) - 1$.*

Now let

$$\tilde{\mathcal{M}}_{R,\epsilon}^{\bar{D}_1}(\tilde{x}_-, \tilde{x}_+) = \bigcup_{(\alpha,t,\tau) \in D', D \leq D' \leq D_1} \tilde{\mathcal{M}}_{R,\epsilon}^{(\alpha,t,\tau)}(\tilde{x}_-, \tilde{x}_+).$$

If we denote n_t and n_τ for the numbers of zero components of the gluing parameter t and τ for a generic $\bar{D}_1 \in \Lambda_{\bar{D}_1}$, where $\bar{D}_1 = \{(\alpha, t, \tau) \mid (\alpha, t, \tau) \in D', D \leq D' \leq D_1\}$, then with similar argument as above or, with more details as Liu-Tian [LT1] proved, we see that for generic choice $\nu \in R$, the moduli space of unparameterized stable (J, H, ν) -connecting orbits

$$\tilde{\mathcal{M}}_{R,\epsilon}^{\bar{D}_1, \nu}(\tilde{x}_-, \tilde{x}_+) \subset \tilde{\mathcal{M}}_{R,\epsilon}^{\bar{D}_1}(\tilde{x}_-, \tilde{x}_+)$$

is a cornered partially smooth pseudomanifold with correct dimension

$$\mu_{rel}(\tilde{x}_-, \tilde{x}_+) - 1 - \Sigma(2n_t + n_\tau).$$

Moreover, the transversality can be achieved for all D' with $D \leq D' \leq D_1$ simultaneously.

⁴Here for simplicity we admit to abuse the notations. Actually, the first summand is the $\bar{\partial}_{J,H}$ -operator for each main component and is the $\bar{\partial}_J$ -operator for each bubble component.

7 Virtual moduli cycle and Floer homology.

7.1 Constructing virtual moduli cycle

Since $PM(\tilde{x}_-, \tilde{x}_+)$ is compact, we can take a finite union of the covering of \mathcal{W} as $\{U_i = U_{\epsilon_{V_i}}(V_i, \mathbf{H}), i = 1, \dots, w\}$, and we use \tilde{U}_i to denote its uniformizer with covering group Γ_i .

Now just as the construction in subsection 4.3, we can define $U_I = \cap_{i \in I} U_i$ where $I = \{1, \dots, w\}$, and the fiber product \tilde{U}_I as in Definition 5.6, so we can get suitable $\mathbf{V}_I \subset U_I$ and $\tilde{\mathbf{V}}_I$ with quotient map $\pi : \tilde{\mathbf{V}}_I \rightarrow \mathbf{V}_I \simeq \tilde{\mathbf{V}}_I/\Gamma_I$ and the projection $\pi_J^I : \tilde{\mathbf{V}}_I \rightarrow \tilde{\mathbf{V}}_J$ for $J \subset I$. Thus as in Definition 5.7 we have a multi-fold atlas $\tilde{\mathcal{V}}$ for \mathcal{W} .

Then as above we have a local orbifold bundle E_i over U_i . For each $[V] \in U_i$, the fiber $\mathcal{F}_i|_{[V]}$ over $[V]$ consists of all elements of $L^{k-1,p}(\Lambda^{0,1}(V^*TM))$, $V \in [V]$ modulo equivalence relation induced by pull-back of sections coming from identification of the domains of V_i , where $\Lambda^{0,1}(V^*TM)$ is the bundle of $(0,1)$ -forms on Σ with respect to the complex structure on Σ and the given compatible almost complex structure J on (M, ω) . Then the local uniformizer \tilde{E}_i of E_i is given by the union of $L^{k-1,p}(\Lambda^{0,1}(\tilde{V}_i^*TM))$, $\tilde{V}_i \in \tilde{U}_i$. The Γ_i also acts on \tilde{E}_i so that $E_i = \tilde{E}_i/\Gamma_i$. In this way, we can reinterpret the $\bar{\partial}_{J,H}$ -operator as a collection of Γ_i -equivalent sections of these local orbifold bundles (E_i, U_i) .

Then as in Lemma 5.3 we can choose subcovers $\{U_i^0\}$ and $\{\mathbf{V}_I\}$ and a suitable partition of unity β_i on \mathcal{W} corresponding to the covering $\{U_i^0\}$. Let $\mathcal{E} = \cup_i E_i$, by the construction of section 4, we obtain a multi-bundle $\tilde{p} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{V}}$.

Denote $R_i = R_{V_i}$, and the projection to the first factor $P : \tilde{U}_i \times R_i \rightarrow \tilde{U}_i$. We can define the section of the pullback bundle $P^*(\tilde{E}_i) \rightarrow \tilde{U}_i \times R_i$ as

$$\iota(i)(\tilde{V}, \nu_i) = \beta_i([V]) \cdot \mathbf{I}_V(\nu_i).$$

Since there are only finite small neighborhoods, we can choose a sufficiently small ϵ such that for all i and generic ν_i , $\bar{\partial}_{J,H} + \iota(i)$ are transverse Fredholm sections of the corresponding pullback bundles. Now we set $R = \oplus_{i=1}^w R_i$ and choose its a small subset $R_\epsilon = \{\nu \in R \mid |\nu_i| \leq \epsilon, \forall i\}$. Then let $\mathcal{W}_\epsilon = \mathcal{W} \times R_\epsilon$, we have a corresponding multi-fold atlas $\tilde{\mathcal{V}}_\epsilon$. Then we can give a multi-bundle structure to $P^*(\tilde{\mathcal{E}}) \rightarrow \tilde{\mathcal{V}}_\epsilon$. Since $\tilde{\mathcal{V}}_\epsilon$ is a good cover, the compatibility condition (8) holds, we get a multi-section \tilde{s} of this multi-bundle as

$$\tilde{s}(\tilde{V}, \nu) = \bar{\partial}_{J,H}(\tilde{V}) + \sum_i \iota(i)(\tilde{V}, \nu_i). \quad (10)$$

It is easy to see that $(\tilde{\mathcal{E}}, \tilde{\mathcal{V}}_\epsilon, \mathcal{W}_\epsilon, \tilde{s})$ is a transverse Fredholm system with index $d = r + \mu_{rel}(\tilde{x}_-, \tilde{x}_+)$ as defined in Definition 5.9. And it follows that when ϵ is small enough for a generic choice of the perturbation $\nu \in R_\epsilon$ the section $\bar{\partial}_{J,H} + \tilde{\nu}$ of $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{V}}$ is Fredholm and have the same zero set as \tilde{s} . Consequently, corresponding to Proposition 6.1, and by using the method of constructing the branched labeled pseudomanifold with boundary Y from the zero set of the multi-section \tilde{s} in subsection 5.3 we have

Proposition 7.1 *For generic $\nu \in R_V$, the multi-section \tilde{s}^ν satisfying $\tilde{s} = P^*(\tilde{s}^\nu)$ is transverse Fredholm and its zero sets \tilde{Z}_I^ν fit together to give a compact branched and labeled pseudomanifold $Y = P\mathcal{M}^\nu(\tilde{x}_-, \tilde{x}_+)$ with boundary $B^\nu(\tilde{x}_-, \tilde{x}_+)$, which is a relative virtual moduli cycle of dimension $d = \mu_{rel}(\tilde{x}_-, \tilde{x}_+) - 1$.*

Proof. The conclusion is straightforward. We only show an explicit proof by Liu-Tian for the compactness of $Y = P\mathcal{M}^\nu(\tilde{x}_-, \tilde{x}_+)$. From the construction in subsection 5.3 we know that $P\mathcal{M}^\nu$ is projected onto $Z_{\mathcal{W}} = \overline{P\mathcal{M}^\nu}$ in \mathcal{W} . Since

$$P\mathcal{M}(\tilde{x}_-, \tilde{x}_+) \subset \cup_{i=1}^w U_i^0,$$

where the U_i^0 is constructed in the Lemma 5.3, we see that $\bar{\partial}_{J,H}$ never becomes zero along the boundary

$$\partial(\cup_{i=1}^w U_i^0) = \overline{\cup_{i=1}^w U_i^0} \setminus \cup_{i=1}^w U_i^0.$$

But $\nu \equiv 0$ along $\partial(\cup_{i=1}^w U_i^0)$, so we have

$$\overline{P\mathcal{M}^\nu} \subset \cup_{i=1}^w U_i^0.$$

Let $\{\bar{V}_i\}_{i=1}^\infty$ be a sequence of $\overline{P\mathcal{M}^\nu}$. We may assume that all \bar{V}_i are contained in \mathbf{V}_I for some $I \in \mathcal{N}$. If we can show that for the corresponding sequence $\{V_i\}_{i=1}^\infty$, with $\bar{V}_i = \pi^I(V_i)$, in $P\mathcal{M}_I^\nu = P\mathcal{M}^\nu \cap \tilde{\mathbf{V}}_I$, all sections ν_I of the bundle $\tilde{E}_I \rightarrow \tilde{\mathbf{V}}_I$ has a uniform bounded $W^{k,p}$ -norm, then by the basic elliptic techniques, there exists a $V_\infty \in \tilde{U}_I$ such that some subsequence of $\{V_i\}_{i=1}^\infty$ is weakly C^∞ -convergent to V_∞ . Then $\overline{P\mathcal{M}^\nu}$ is compact.

Since $\nu = \sum_{i,j} a_{ij} e_{ij}$ with $\{e_{ij}; j = 1, \dots, n_i\}$ being the basis of R_i , and $\nu \in R_\epsilon$, we see that $|a_{ij}|$ are bounded. So we only need to prove that all the lifting $\{e_{ij}\}_I$ over $\tilde{\mathbf{V}}_I$ of e_{ij} , which is defined over \tilde{U}_i originally, are still bounded. We still only consider the case $i \in I$. Actually, if we can prove that all changes of coordinates between \tilde{U}_i 's are induced from those reparametrizations that stay inside a compact subset of G_Σ , then the boundedness of $\|\nu_I\|_{k,p}$ will follow.

Now we denote the closure of U_i by U_i^c , $i = 1, \dots, w$, and $U_{ij}^c = U_i^c \cap U_j^c$. Let \tilde{U}_i^c and $\tilde{U}_{ij}^c \subset \tilde{U}_i^c$ be the lifting of them in the uniformizer \tilde{U}_i . Then we denote the compact set

$$P\mathcal{M}_{ij} = P\mathcal{M}(\tilde{x}_-, \tilde{x}_+) \cap U_{ij}^c.$$

Let

$$\{Z_k^{ij} \mid Z_k^{ij} \subset \mathcal{B}(\tilde{x}_-, \tilde{x}_+), k = 1, \dots, m^{ij}\}$$

be an open covering of $P\mathcal{M}_{ij}$ in $\mathcal{B}(\tilde{x}_-, \tilde{x}_+)$ such that each component of $\pi_i^{-1}(Z_k^{ij})$ and $\pi_j^{-1}(Z_k^{ij})$ in \tilde{U}_i^c and \tilde{U}_j^c is a uniformizer of Z_k^{ij} , respectively. Now for each fixed pair of components of $\pi_i^{-1}(Z_k^{ij})$ and $\pi_j^{-1}(Z_k^{ij})$, the equivalence between them are induced by some automorphisms of domain which are contained in a compact subset of $\prod G_\Sigma$. Then let $Z^{ij} = \cup_k Z_k^{ij}$ and for each $i = 1, \dots, w$, replace U_i by

$$(U_i \setminus \cup_{k \neq i} U_{ik}^c) \bigcup_{k \neq i} Z^{ik}.$$

They still form an open covering of $P\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ and all previous constructions still work. Now all changes of coordinates are induced by a compact subset. \square

Recall that each top stratum S of Y lying in the image $q_I : \tilde{Y}_I \rightarrow Y$ is associated a positive rational label λ_I . They can fit together to give each δ_i -oriented top component M_i a label λ_i . Then we can define a rational number for the virtual moduli space

$$\#(Y) = \#(P\mathcal{M}^\nu) = \sum_i \delta_i \lambda_i.$$

We say any compact branched and labeled pseudomanifold $P\mathcal{M}^\nu(\tilde{x}_-, \tilde{x}_+)$ as above constructed is the *regularization* or *virtual moduli space* of the stable moduli space $P\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$. In particular, we care about such $d = \mu_{rel}(\tilde{x}_-, \tilde{x}_+) - 1 = 0$ and 1 dimensional pseudomanifolds

$$P\mathcal{M}^\nu(\tilde{x}_i, \tilde{x}_{i+1}) \quad \text{and} \quad P\mathcal{M}^\nu(\tilde{x}_i, \tilde{x}_{i+2}).$$

The former is obviously a finite set. And the latter is a branched and labeled 1-pseudomanifold with boundary which, via Floer's gluing method, consists of pairs $[V \# U]$ with $[V] \in P\mathcal{M}^\nu(\tilde{x}_i, \tilde{x}_{i+1})$ and $[U] \in P\mathcal{M}^\nu(\tilde{x}_{i+1}, \tilde{x}_{i+2})$. Recall that in section 5 we can associate to each boundary point a rational number $\rho([V \# U])$, then from the Lemma 5.2 we see that the total oriented number of its boundary is $\#(B^\nu(\tilde{x}_i, \tilde{x}_{i+2})) = \sum_{x \in B^\nu} \rho(x) = 0$. Thus, we naturally have the following result

Corollary 7.1 1° . *If the relative index*

$$\mu_{rel}(\tilde{x}_-, \tilde{x}_+) = \mu_{CZ}(\tilde{x}_+) - \mu_{CZ}(\tilde{x}_-) = 1,$$

then $P\mathcal{M}^\nu(\tilde{x}_-, \tilde{x}_+)$ is a finite set;

2° . *If $\mu_{rel}(\tilde{x}_-, \tilde{x}_+) = 2$, then in the sense of partially smooth category, the oriented boundary $B^\nu(\tilde{x}_-, \tilde{x}_+) = \partial(P\mathcal{M}^\nu(\tilde{x}_-, \tilde{x}_+))$ is a finite set with the total oriented number $\#(B^\nu(\tilde{x}_-, \tilde{x}_+)) = 0$. Moreover,*

$$\partial(P\mathcal{M}^\nu(\tilde{x}_-, \tilde{x}_+)) = \sum_{\mu_{rel}(\tilde{x}_-, \tilde{y})=1; \mu_{rel}(\tilde{y}, \tilde{x}_+)=1} P\mathcal{M}^\nu(\tilde{x}_-, \tilde{y}) \times P\mathcal{M}^\nu(\tilde{y}, \tilde{x}_+).$$

Remark. This Corollary can be generalized to the case for t -dependent pair (J_t, H_t) that will be used in the continuation argument.

7.2 Define the Floer-type homology

We come to define the Floer chain complex. As the usual way, we first define a graded \mathbb{Q} -space $C_* = C_*(J, H, \phi) = \bigoplus_n C_n(J, H, \phi)$ as follows. Recall that $F = F_H$ is the functional defined in section 2 and $Crit(F)$ is the set of all critical points. We denote by $Crit_n(F)$ the subset of all $\tilde{x} \in Crit(F)$ with Conley-Zehnder index $\mu_{CZ}(\tilde{x}) = n$. Let $C_n = C_n(J, H, \phi)$ be the set of all formal sums

$$\xi = \sum_{\tilde{x} \in Crit_n(F)} \xi_{\tilde{x}} \tilde{x},$$

where $\xi_{\tilde{x}} \in \mathbb{Q}$ such that for any constant $c > 0$

$$\#\{\tilde{x} \in Crit_n(F) \mid \xi_{\tilde{x}} \neq 0, F(\tilde{x}) \geq c\} < \infty.$$

Thus C_* is an infinite dimensional vector space over \mathbb{Q} in general, however, if we introduce the so-called Novikov ring $\Lambda_{\omega,\phi}$ (which is a field here) as follows, we will see that it is a finite dimensional vector space over $\Lambda_{\omega,\phi}$.

Recall that Γ is the covering group introduced in section 2, and we assume that there is an injective homomorphism $i : \Gamma \rightarrow \pi_2(M)$. Then the function $\phi_\omega : \pi_2(M) \rightarrow \mathbb{R}$, $A \mapsto \int_A \omega$, induced a *weight homomorphism* $\phi : \Gamma \rightarrow \mathbb{R}$ which is injective. Then Hofer-Salamon [HS] showed that the group Γ is isomorphic to a free abelian group with finite many generators. We suppose $\{e_1, \dots, e_k\}$ be the basis of Γ , so for any $A \in \Gamma$, we have $A = \sum_{i=1}^k A_i e_i$. Let $t = (t_1, \dots, t_k)$, then we denote t^A for $\prod_{i=1}^k t_i^{A_i}$. Then the Novikov ring $\Lambda_{\omega,\phi}$ is the set of formal sums as

$$\lambda = \sum_{A \in \Gamma} \lambda_A t^A,$$

where $\lambda_A \in \mathbb{Q}$ such that for any constant $c > 0$

$$\#\{A \in \Gamma \mid \lambda_A \neq 0, \phi(A) \leq c\} < \infty.$$

Since the coefficient is the rational field \mathbb{Q} , our Novikov ring is also a field. We note that the multiplication in Novikov ring is

$$\lambda \cdot \mu = \sum_{A, B \in \Gamma} \lambda_A \cdot \mu_B t^{A+B}.$$

It is easy to verify that the following defined *scalar product*

$$\lambda \cdot \xi = \sum_{\tilde{x} \in \text{Crit}(F)} \left(\sum_{A \in \Gamma} \lambda_A \cdot \xi_{(-A) \# \tilde{x}} \right) \tilde{x}$$

is still in C_* , where the connect sum $(-A) \# \tilde{x}$ is induced from the $\pi_2(M)$ action on $\tilde{\Omega}_\phi$. Then we can consider the space C_* as a finite dimensional vector space over the field $\Lambda_{\omega,\phi}$ with dimension of $\#\text{Fix}(\phi_H)$.

Since from the first part of Corollary 7.1 we know that when $\mu_{rel}(\tilde{y}, \tilde{x}) = \mu_{CZ}(\tilde{x}) - \mu_{CZ}(\tilde{y}) = 1$, the number $\#(PM^\nu(\tilde{y}, \tilde{x}))$ is finite, then we just define the boundary operator $\delta_\nu : C_* \rightarrow C_*$ as: for any $\tilde{x} \in \text{Crit}_n(F)$

$$\delta_\nu(\tilde{x}) = \sum_{\tilde{y} \in \text{Crit}_{n-1}(F)} \#(PM^\nu(\tilde{y}, \tilde{x})) \tilde{y}.$$

Then by the conclusion in the second part of Corollary 7.1, we know that for any two \tilde{z}, \tilde{x} with $\mu_{rel}(\tilde{z}, \tilde{x}) = 2$, the oriented number

$$\#[\partial(PM^\nu(\tilde{z}, \tilde{x}))] = \sum_{\mu_{rel}(\tilde{z}, \tilde{y})=1; \mu_{rel}(\tilde{y}, \tilde{x})=1} \#(PM^\nu(\tilde{z}, \tilde{y})) \times \#(PM^\nu(\tilde{y}, \tilde{x})) = 0.$$

So we have for any $\tilde{x} \in \text{Crit}_n(F)$,

$$\delta_\nu^2(\tilde{x}) = \sum_{\mu_{CZ}(\tilde{z})=n-2} \left[\sum_{\mu_{CZ}(\tilde{y})=n-1} \#(PM^\nu(\tilde{z}, \tilde{y})) \times \#(PM^\nu(\tilde{y}, \tilde{x})) \right] \tilde{z} = 0.$$

Thus, we just define the Floer homology associated to (J, H, ν) of the symplectic manifold (M, ω) and a symplectomorphism ϕ as the homology of the chain complex (C_*, δ_ν) , denoted by $FH_*(J, H, \phi, \nu)$ or $FH_*(J, H, \phi)$.

As the last step, we should prove that the Floer homology groups $FH_*(J, H, \phi, \nu)$ are independent of the almost complex structure J_t and the time-dependent Hamiltonian H used to define them. To do this, we need a continuation argument as the standard method used in [F3][SZ][LT1], *etc.*

Given a fixed symplectomorphism ϕ and two suitable triples (J_0, H_0, ν_0) and (J_1, H_1, ν_1) which are fine in the construction as above. We want to show that

$$FH_*(J_0, H_0, \nu_0) \cong FH_*(J_1, H_1, \nu_1).$$

Thus, we need to show there exists a chain homotopy. Firstly, we define a chain homomorphism

$$\Phi_1^0 : (C_*(H_0), \delta_{J_0, H_0, \nu_0}) \rightarrow (C_*(H_1), \delta_{J_1, H_1, \nu_1}).$$

Suppose we have a family of generic pairs (J_s, H_s) , $s \in \mathbb{R}$, so that $(J_s, H_s) \equiv (J_0, H_0)$ for $s \leq 0$ and $(J_s, H_s) \equiv (J_1, H_1)$ for $s \geq 1$. For two critical points in different spaces

$$\tilde{x}_0 \in \widetilde{\text{Fix}}(\phi_{H_0}) \subset \widetilde{\Omega}_\phi(H_0) \quad \text{and} \quad \tilde{x}_1 \in \widetilde{\text{Fix}}(\phi_{H_1}) \subset \widetilde{\Omega}_\phi(H_1),$$

we can similarly as before define the moduli space

$$P\mathcal{M}(J_s, H_s, \tilde{x}_0, \tilde{x}_1)$$

of stable continuation trajectories, which consists of stable (J_s, H_s) -orbits connecting \tilde{x}_0 and \tilde{x}_1 , say the element is

$$V = ((v_1, \dots, v_K), (f_1, \dots, f_l), o) : \Sigma \rightarrow M, \quad v_j = u_j|_{\mathbb{R} \times [0, 1]},$$

with some differences in that on each main component Σ_m , the map $u_m : \Sigma_m \rightarrow M$, $m = 1, \dots, K$, satisfies the following equation

$$\bar{\partial}_{J_s, H_s}(u_m) = \frac{\partial u_m}{\partial s}(s, t) + J_{s, t}(u_m(s, t))\left(\frac{\partial u_m}{\partial t}(s, t) - X_t(H_s)\right) = 0. \quad (11)$$

And we require that there is a $m_0 \in \{1, \dots, K\}$ so that when $m < m_0$, u_m is a stable (J_0, H_0) -map and when $m > m_0$ it is a stable (J_1, H_1) -map satisfying the equation (1).

We can apply all above construction to this new stable moduli space, for example, we can similarly define the ambient space $\mathcal{B}(J_s, H_s, \tilde{x}_0, \tilde{x}_1)$, the neighborhood \mathcal{W} and the compact virtual moduli space $P\mathcal{M}^{\nu_s}(J_s, H_s, \tilde{x}_0, \tilde{x}_1)$, *etc.* Since for those stable (J_s, H_s) -orbits the s -invariance does not hold for the distinct main component u_{m_0} , the dimension of $P\mathcal{M}^{\nu_s}(\tilde{x}_0, \tilde{x}_1)$ will be

$$\mu_{rel}(\tilde{x}_0, \tilde{x}_1) = \mu_{CZ}(H_1, \tilde{x}_1) - \mu_{CZ}(H_0, \tilde{x}_0).$$

So when $\mu_{CZ}(H_1, \tilde{x}_1) = \mu_{CZ}(H_0, \tilde{x}_0)$, $P\mathcal{M}^{\nu_s}(\tilde{x}_0, \tilde{x}_1)$ is a finite set with well-defined rational oriented number $\#(P\mathcal{M}^{\nu_s}(\tilde{x}_0, \tilde{x}_1))$. Consequently, we can define

the homomorphism between the spaces with same grade $C_n(H_0) \rightarrow C_n(H_1)$ (then it is also a homomorphism between $C_*(H_0)$ and $C_*(H_1)$) as

$$\Phi_1^0(\tilde{x}_0) = \sum_{\mu_{CZ}(\tilde{x}_1)=n} \#(PM^{\nu_s}(J_s, H_s, \tilde{x}_0, \tilde{x}_1))\tilde{x}_1,$$

for $\tilde{x}_0 \in C_n(H_0)$.

Similarly, we can define a chain homomorphism

$$\Phi_0^1 : (C_*(H_1), \delta_{J_1, H_1, \nu_1}) \rightarrow (C_*(H_0), \delta_{J_0, H_0, \nu_0}).$$

Then just as the classical method of introducing an extra parameter ρ and applying all constructions as before to the two parameters family (J_s^ρ, H_s^ρ) , used by Floer [F1]-[F3] with the modification in that we replace the classical moduli space of (J_s, H_s) -orbits $\mathcal{M}(J_s, H_s)$ by the branched labeled pseudomanifold, *i.e* the virtual moduli space $PM^{\nu_s}(J_s^\rho, H_s^\rho)$ and correspondingly we replace $\mathcal{M}(J_s^\rho, H_s^\rho)$ by $PM^{\nu_s^\rho}(J_s^\rho, H_s^\rho)$, we can prove that there exist chain homotopies

$$\Phi_0^1 \circ \Phi_1^0 \sim Id_{C_*(H_0)} \quad \text{and} \quad \Phi_1^0 \circ \Phi_0^1 \sim Id_{C_*(H_1)}.$$

Therefore, we have induced an isomorphism

$$(\Phi_1^0)^* : FH_*(J_0, H_0, \nu_0) \rightarrow FH_*(J_1, H_1, \nu_1).$$

We omit the details, and refer the reader to [F3][LT1][HS][SZ], *etc.* for similar arguments with respect to establishing isomorphism of Floer (co)homologies for Hamiltonian periodic solutions. So we can just denote the Floer homology associated with a compact symplectic manifold (M, ω) and a symplectomorphism ϕ by $FH_*(M, \omega, \phi, \nu)$ or simply by $FH_*(\phi, \nu)$.

Although $FH_*(\phi, \nu)$ are in general dependent of the symplectomorphism ϕ , we will show that they only depend on ϕ up to Hamiltonian isotopy. That is to say, there is a natural isomorphism

$$FH_*(\phi_0) \rightarrow FH_*(\phi_1)$$

if ϕ_0 and ϕ_1 are related by a Hamiltonian isotopy. Let

$$\phi_t = \varphi_t^{-1} \circ \phi_0$$

be a Hamiltonian isotopy from ϕ_0 to $\phi_1 = \varphi_1^{-1} \circ \phi_0$. That is to say, there exist t -dependent Hamiltonian function G_t and Hamiltonian vector field Y_t on M satisfying

$$\frac{d}{dt}\varphi_t = Y_t \circ \varphi_t, \quad \iota(Y_t)\omega = dG_t.$$

Recall for symplectomorphism $\phi_0 = \phi$ satisfying $H_t = H_{t+1} \circ \phi_0$, we have a map $u : \mathbb{R}^2 \rightarrow M$ which is the solution of the equation (1) with boundary condition (2) and limits (3). Then we define a new map $v : \mathbb{R}^2 \rightarrow M$ so that

$$v(s, t) = \varphi_t^{-1}(u(s, t)), \quad J'_t = \varphi_t^* J_t. \tag{12}$$

We see that $\phi_1^* J'_{t+1} = J_t$ and $v(s, t)$ is the solution of the following equation

$$\frac{\partial v}{\partial s} + J'_t(v) \left(\frac{\partial v}{\partial t} - \varphi_t^{-1} \circ (X_t - Y_t) \circ \varphi_t(v) \right) = 0 \quad (13)$$

with boundary condition

$$v(s, t+1) = \phi_1(v(s, t)). \quad (14)$$

and the solution has limits

$$\lim_{s \rightarrow \pm\infty} v(s, t) = \varphi_t^{-1} \psi_t(x_{\pm}), \quad x_{\pm} = \phi_H x_{\pm}, \quad (15)$$

where recall x_{\pm} are nondegenerate fixed points of $\phi_H = \psi_1^{-1} \circ \phi$. Thus there is a one-to-one correspondence between the solutions of (1), (2) and the solutions of (13), (14). We can extend the arguments above to the virtual case in a similar way to get a 1-1 correspondence between their related virtual moduli space PM^{ν_0} and PM^{ν_1} . So we can obtain the isomorphism $FH_*(\phi_0) \simeq FH_*(\phi_1)$.

In particular, as what Dostoglou-Salamon claimed in [DS2], by the “fixed point index” property of the Maslov index listed in section 2, the Euler characteristic is just the Lefschetz number of ϕ

$$\chi(FH_*(\phi)) = \sum_{x \in \text{Fix}(\phi_H)} \text{sign det}(Id - d\phi_H(x)) = L(\phi).$$

Remark. Recall the construction of Floer homology is related to the choices of those based paths γ_0 in each connected component of Ω_{ϕ} . In fact, we should denote the Floer homology by $FH_*(\phi, \gamma_0)$. Nevertheless, we can show that our construction above depends only on the homotopy class of those based paths. A similar argument can be found in [FO³].

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